Eigenvalues and Eigenvectors

Initialization

```
ClearAll["Global`*"];
Off[General::spell, General::spell1];
```

Eigenvalues and Eigenvectors

We will now review some ideas from linear algebra. Proofs of the theorems are either left as exercises or can be found in any standard text on linear algebra. We know how to solve *n* linear equations in *n* unknowns. It was assumed that the determinant of the matrix was nonzero and hence that the solution was unique. In the case of a homogeneous system $\boldsymbol{AX} = \mathbf{0}$, if $\det(a) \neq 0$, the unique solution is the trivial solution $\boldsymbol{X} = \mathbf{0}$. If $\det(a) = 0$, there exist nontrivial solutions to $\boldsymbol{AX} = \mathbf{0}$. Suppose that $\det(a) = 0$, and consider solutions to the homogeneous linear system

$$a_{1,1} x_1 + a_{1,2} x_2 + a_{1,3} x_3 + \cdots + a_{1,n-1} x_{n-1} + a_{1,n} x_n = 0$$
$$a_{2,1} x_1 + a_{2,2} x_2 + a_{2,3} x_3 + \cdots + a_{2,n-1} x_{n-1} + a_{2,n} x_n = 0$$
$$a_{3,1} x_1 + a_{3,2} x_2 + a_{3,3} x_3 + \cdots + a_{3,n-1} x_{n-1} + a_{3,n} x_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \quad \vdots$$
$$a_{n,1} x_1 + a_{n,2} x_2 + a_{n,3} x_3 + \cdots + a_{n,n-1} x_{n-1} + a_{n,n} x_n = b_n$$

A homogeneous system of equations always has the trivial solution $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$. Gaussian elimination can be used to obtain the reduced row echelon form which will be used to form a set of relationships between the variables, and a non-trivial solution.

Example 1. Find the nontrivial solutions to the homogeneous system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ 5x_1 + 4x_2 + x_3 &= 0 \end{aligned}$$

Solution 1.

Use Gaussian elimination to eliminate x_1 and the result is

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \\ -6x_2 + 6x_3 &= 0 \end{aligned}$$

Since the third equation is a multiple of the second equation, this system reduces to two equations in three unknowns:

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}$$

We can select one unknown and use it as a parameter. For instance, let $x_3 = t$; then the second equation implies that $x_2 = t$and the first equation is used to compute $x_1 = t$. Therefore, the solution can be expressed as the set of relations:

$$\begin{aligned} x_1 &= -t \\ x_2 &= t \\ x_3 &= t \end{aligned}$$

or

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

We can fin the solution by entering the equations into *Mathematica*.

```
eqns = {  x1 + 2 x2 - x3 == 0,
          2 x1 +   x2 + x3 == 0,
          5 x1 + 4 x2 + x3 ==  0 };
```

Identify the matrix of coefficients **A** and column vector **B** for the matrix problem **AX** = **B**.

```
vars = {x1 , x2 , x3};
Needs["LinearAlgebra`MatrixManipulation`"];
mats = LinearEquationsToMatrices[eqns, vars];
A = mats[[1]];
B = mats[[2]];
Print["Solve the equations"];
Print[TableForm[eqns]];
Print["A = ", MatrixForm[A]];
Print["B = ", MatrixForm[B]];
Print["Solve the equation A X = B"];
Print[" A  = ", MatrixForm[A],
  MatrixForm[vars], " = ", MatrixForm[B]];
Print["|A| = ", Det[A]];
Print["Hence the system will have non-trivial solutions."];
```

Developer`LinearExpressionToMatrix::obs :
Developer`LinearExpressionToMatrix has been superseded by CoefficientArrays, and is now obsolete. It will not be included in *Mathematica* version 8.

Solve the equations

$x_1 + 2 x_2 - x_3 == 0$
$2 x_1 + x_2 + x_3 == 0$
$5 x_1 + 4 x_2 + x_3 == 0$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve the equation A X = B

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$|A| = 0$

Hence the system will have non-trivial solutions.

Form the augmented matrix **M = [A, B]** and perform Gauss-Jordan elimination with row interchanges.

```
b = Partition[B, 1];
M = AppendRows[A, b]; m = M;
Print["A = ", MatrixForm[A]];
Print["B = ", MatrixForm[B]];
Print["The augmented matrix M = [A,B] is"];
Print["M = ", MatrixForm[M]];
```

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$$

Find the reduced row echelon form of the augmented matrix **M = [A, B]**.

```
Print["The augmented matrix M = [A,B] is"];
Print["M = ", MatrixForm[M]];
```

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$$

```
M[[2]] = M[[2]] - 2 M[[1]];
M[[3]] = M[[3]] - 5 M[[1]];
Print[MatrixForm[M]];
```

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{pmatrix}$$

```
M[[2]] = -(1/3) M[[2]];
Print[MatrixForm[M]];
```

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -6 & 6 & 0 \end{pmatrix}$$

```
M[[1]] = M[[1]] - 2 M[[2]];
M[[3]] = M[[3]] + 6 M[[2]];
Print[MatrixForm[M]];
```

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equation form for this matrix is

$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

There is one free variable which we choose to be $x_3 = t$. It is used in computing $x_2 = t$ and $x_1 = -t$.

The solution vector $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is $X = \begin{pmatrix} -t \\ t \\ t \end{pmatrix}$;

We are done.

Aside. We can verify that this is the solution by direct multiplication **A X**. This is just for fun !

```
Print["Solve the equation A X = 0"];
Print["  A = ",
  MatrixForm[A] MatrixForm[vars], " = ", MatrixForm[B]];
Print["  X = ", MatrixForm[vars], " = ", MatrixForm[X]];
Print["Does  ", MatrixForm[A],
  MatrixForm[X], " = ", MatrixForm[B], " ?"];
Print["      ", MatrixForm[A.X], " = ", MatrixForm[B]];
Print[Flatten[A.X] == Flatten[B]];
```

Solve the equation A X = 0

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = X$$

$$\text{Does} \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ?$$

$$\{\{1, 2, -1\}, \{2, 1, 1\}, \{5, 4, 1\}\}.X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

{{1, 2, -1}, {2, 1, 1}, {5, 4, 1}}.X == {0, 0, 0}

Aside. We can let *Mathematica* find the reduced row echelon matrix. This is just for fun !

```
M = m;
Print["M = ", MatrixForm[M]];
Print["The row reduced echelon form is"];
Print["    ", MatrixForm[RowReduce[M]]];
```

$$M = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Background for Eigenvalues and Eigenvectors

Definition (Linearly Independent). The vectors $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n$

are said to be linearly independent if the equation

$$c_1 \mathbf{U}_1 + c_2 \mathbf{U}_2 + \ldots + c_n \mathbf{U}_n = 0$$

implies that $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. If the vectors are not linearly independent they are said to be linearly dependent.

Two vectors in $\mathfrak{R}^2$ are linearly independent if and only if they are not parallel. Three vectors in $\mathfrak{R}^3$ are linearly independent if and only if they do not lie in the same plane.

Definition (Linearly Dependent). The vectors $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n$ are said to be linearly dependent if there exists a set of numbers $\{c_1, c_2, \ldots, c_n\}$ not all zero, such that

$$c_1 \mathbf{U}_1 + c_2 \mathbf{U}_2 + \ldots + c_n \mathbf{U}_n = 0.$$

Theorem. The vectors $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n$ are linearly dependent if and only if at least one of them is a linear combination of the others.

A desirable feature for a vector space is the ability to express each vector as s linear combination of vectors chosen from a small subset of vectors. This motivates the next definition.

Definition (Basis). Suppose that $S = \{\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_m\}$ is a set of m vectors in $\mathfrak{R}^n$. The set S i s called a basis for $\mathfrak{R}^n$ if for every vector $\mathbf{X} \in \mathfrak{R}^n$ there exists a unique set

of scalars $\{c_1, c_2, \ldots, c_m\}$ so that $\mathbf{X}$ can be expressed as the linear combination

$$\mathbf{X} = c_1 \mathbf{U}_1 + c_2 \mathbf{U}_2 + \ldots + c_n \mathbf{U}_n$$

Theorem. In $\mathbb{R}^n$, any set of n linearly independent vectors forms a basis of $\mathbb{R}^n$. Each vector $\mathbf{X} \in \mathbb{R}^n$ is uniquely expressed as a linear combination of the basis vectors.

Theorem. Let $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ be vectors in $\mathbb{R}^n$.

(i) If $m > n$, then the vectors are linearly independent.

(ii) If $m = n$, then the vectors are linearly dependent if and only if $\det(\mathbf{K}) = 0$, where $\mathbf{K} = [\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m]$.

Proof **Eigenvalues and Eigenvectors** **Eigenvalues and Eigenvectors**

Applications of mathematics sometimes encounter the following questions: What are the singularities of $\mathbf{A} - \lambda\mathbf{I}$, where λ is a parameter? What is the behavior of the sequence of vectors $\{\mathbf{A}^j \mathbf{X}_0\}_{j=0}^{\infty}$? What are the geometric features of a linear transformation? Solutions for problems in many different disciplines, such as economics, engineering, and physics, can involve ideas related to these equations. The theory of eigenvalues and eigenvectors is powerful enough to help solve these otherwise intractable problems.

Let $\mathbf{A}$ be a square matrix of dimension $n \times n$ and let $\mathbf{X}$ be a vector of dimension n. The product $\mathbf{Y} = \mathbf{A}\mathbf{X}$ can be viewed as a linear transformation from n-dimensional space into itself. We want to find scalars λ for which there exists a nonzero vector $\mathbf{X}$ such that

$$(1) \qquad \mathbf{A}\mathbf{X} = \lambda\mathbf{X};$$

that is, the linear transformation $T(\mathbf{X}) = \mathbf{A}\mathbf{X}$ maps $\mathbf{X}$ onto the multiple $\lambda\mathbf{X}$. When this occurs, we call $\mathbf{X}$ an eigenvector that corresponds to the eigenvalue λ, and together they form the eigenpair λ, **A** for **A**. In general, the scalar λ and vector **X** can involve complex numbers. For simplicity, most of our illustrations will involve real calculations. However, the techniques are easily extended to the complex case. The n × n identity matrix **I** can be used to write equation (1) in the form

(2) $$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = 0.$$

The significance of equation (2) is that the product of the matrix $\mathbf{A} - \lambda \mathbf{I}$ and the nonzero vector **X** is the zero vector! The theorem of homogeneous linear system says that (2) has nontrivial solutions if and only if the matrix $\mathbf{A} - \lambda \mathbf{I}$ is singular, that is,

(3) $$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

This determinant can be written in the form

(4)

$$\begin{vmatrix} a_{1,1} - \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} - \lambda & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} - \lambda & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} - \lambda & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n} - \lambda \end{vmatrix} = 0$$

Definition (Characteristic Polynomial). When the determinant in (4) is expanded, it becomes a polynomial of degree n, which is called the characteristic polynomial

(5)

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$$

$$p(\lambda) = (-1)^n \left(\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-2} \lambda^2 + c_{n-1} \lambda + c_n\right)$$

Exploration for p(λ)

```
n = 2;
A = Table[a_{i,j}, {i, n}, {j, n}];
I_n = IdentityMatrix[n];
p[λ_] = Det[A - λ I_n];
solset = Solve[p[λ] == 0, λ];
Print["  A  = ", MatrixForm[A]];
Print["p[λ] = Det[", MatrixForm[A - λ I_n], "]"];
Print["p[λ] = ",
  Collect[Expand[p[λ] - p[0]], λ], "+(", Expand[p[0]], ")"];
Print["Solve p[λ] = 0 and get"];
Print[TableForm[solset]];
```

A = $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$

p[λ] = Det[$\begin{pmatrix} -\lambda + a_{1,1} & a_{1,2} \\ a_{2,1} & -\lambda + a_{2,2} \end{pmatrix}$]

p[λ] = $\lambda^2 + \lambda\,(-a_{1,1} - a_{2,2}) + (-a_{1,2}\,a_{2,1} + a_{1,1}\,a_{2,2})$

Solve p[λ] = 0 and get

$\lambda \to \frac{1}{2}\left(a_{1,1} + a_{2,2} - \sqrt{a_{1,1}^2 + 4\,a_{1,2}\,a_{2,1} - 2\,a_{1,1}\,a_{2,2} + a_{2,2}^2}\right)$

$\lambda \to \frac{1}{2}\left(a_{1,1} + a_{2,2} + \sqrt{a_{1,1}^2 + 4\,a_{1,2}\,a_{2,1} - 2\,a_{1,1}\,a_{2,2} + a_{2,2}^2}\right)$

```
n = 3;
A = Table[a_{i,j}, {i, n}, {j, n}];
I_n = IdentityMatrix[n];
p[λ_] = Det[A - λ I_n];
Print["  A  = ", MatrixForm[A]];
Print["p[λ] = Det[", MatrixForm[A - λ I_n], "]"];
Print["p[λ] = ",
  Collect[Expand[p[λ] - p[0]], λ], "+(", Expand[p[0]], ")"];
```

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$$p[\lambda] = \text{Det}\left[\begin{pmatrix} -\lambda + a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & -\lambda + a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & -\lambda + a_{3,3} \end{pmatrix}\right]$$

$$p[\lambda] = -\lambda^3 + \lambda^2 (a_{1,1} + a_{2,2} + a_{3,3}) + \lambda (a_{1,2} a_{2,1} - a_{1,1} a_{2,2} + a_{1,3} a_{3,1} + a_{2,3} a_{3,2} - a_{1,1} a_{3,3} - a_{2,2} a_{3,3}) + (-a_{1,3} a_{2,2} a_{3,1} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3} + a_{1,1} a_{2,2} a_{3,3})$$

```
n = 4;
A = Table[a_{i,j}, {i, n}, {j, n}];
I_n = IdentityMatrix[n];
p[λ_] = Det[A - λ I_n];
Print["   A   = ", MatrixForm[A]];
Print["p[λ]  = Det[", MatrixForm[A - λ I_n], "]"];
Print["p[λ]  = ",
  Collect[Expand[p[λ] - p[0]], λ], "+(", Expand[p[0]], ")"];
```

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

$$p[\lambda] = \text{Det}\left[\begin{pmatrix} -\lambda + a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & -\lambda + a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & -\lambda + a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & -\lambda + a_{4,4} \end{pmatrix}\right]$$

$$\begin{aligned} p[\lambda] = {} & \lambda^4 + \lambda^3 (-a_{1,1} - a_{2,2} - a_{3,3} - a_{4,4}) + \\ & \lambda^2 (-a_{1,2} a_{2,1} + a_{1,1} a_{2,2} - a_{1,3} a_{3,1} - a_{2,3} a_{3,2} + a_{1,1} a_{3,3} + a_{2,2} a_{3,3} - \\ & \quad a_{1,4} a_{4,1} - a_{2,4} a_{4,2} - a_{3,4} a_{4,3} + a_{1,1} a_{4,4} + a_{2,2} a_{4,4} + a_{3,3} a_{4,4}) + \\ & \lambda (a_{1,3} a_{2,2} a_{3,1} - a_{1,2} a_{2,3} a_{3,1} - a_{1,3} a_{2,1} a_{3,2} + a_{1,1} a_{2,3} a_{3,2} + a_{1,2} a_{2,1} a_{3,3} - \\ & \quad a_{1,1} a_{2,2} a_{3,3} + a_{1,4} a_{2,2} a_{4,1} - a_{1,2} a_{2,4} a_{4,1} + a_{1,4} a_{3,3} a_{4,1} - \\ & \quad a_{1,3} a_{3,4} a_{4,1} - a_{1,4} a_{2,1} a_{4,2} + a_{1,1} a_{2,4} a_{4,2} + a_{2,4} a_{3,3} a_{4,2} - a_{2,3} a_{3,4} a_{4,2} - \\ & \quad a_{1,4} a_{3,1} a_{4,3} - a_{2,4} a_{3,2} a_{4,3} + a_{1,1} a_{3,4} a_{4,3} + a_{2,2} a_{3,4} a_{4,3} + a_{1,2} a_{2,1} a_{4,4} - \\ & \quad a_{1,1} a_{2,2} a_{4,4} + a_{1,3} a_{3,1} a_{4,4} + a_{2,3} a_{3,2} a_{4,4} - a_{1,1} a_{3,3} a_{4,4} - a_{2,2} a_{3,3} a_{4,4}) \\ & + (a_{1,4} a_{2,3} a_{3,2} a_{4,1} - a_{1,3} a_{2,4} a_{3,2} a_{4,1} - a_{1,4} a_{2,2} a_{3,3} a_{4,1} + \\ & \quad a_{1,2} a_{2,4} a_{3,3} a_{4,1} + a_{1,3} a_{2,2} a_{3,4} a_{4,1} - \\ & \quad a_{1,2} a_{2,3} a_{3,4} a_{4,1} - a_{1,4} a_{2,3} a_{3,1} a_{4,2} + a_{1,3} a_{2,4} a_{3,1} a_{4,2} + \\ & \quad a_{1,4} a_{2,1} a_{3,3} a_{4,2} - a_{1,1} a_{2,4} a_{3,3} a_{4,2} - a_{1,3} a_{2,1} a_{3,4} a_{4,2} + \\ & \quad a_{1,1} a_{2,3} a_{3,4} a_{4,2} + a_{1,4} a_{2,2} a_{3,1} a_{4,3} - a_{1,2} a_{2,4} a_{3,1} a_{4,3} - \\ & \quad a_{1,4} a_{2,1} a_{3,2} a_{4,3} + a_{1,1} a_{2,4} a_{3,2} a_{4,3} + a_{1,2} a_{2,1} a_{3,4} a_{4,3} - \\ & \quad a_{1,1} a_{2,2} a_{3,4} a_{4,3} - a_{1,3} a_{2,2} a_{3,1} a_{4,4} + a_{1,2} a_{2,3} a_{3,1} a_{4,4} + \\ & \quad a_{1,3} a_{2,1} a_{3,2} a_{4,4} - a_{1,1} a_{2,3} a_{3,2} a_{4,4} - a_{1,2} a_{2,1} a_{3,3} a_{4,4} + a_{1,1} a_{2,2} a_{3,3} a_{4,4}) \end{aligned}$$

There exist exactly n roots (not necessarily distinct) of a polynomial of degree n. Each root λ can be substituted into equation (3) to obtain an underdetermined system of equations that has a corresponding nontrivial solution vector **X**. If λ is real, a real eigenvector **X** can be constructed. For emphasis, we state the following definitions.

Definition (Eigenvalue). If **A** is and $n \times n$ real matrix, then its n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the real and complex roots of the characteristic polynomial

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}).$$

Definition (Eigenvector). If λ is an eigenvalue of $\mathbf{A}$ and the nonzero vector $\mathbf{V}$ has the property that

$$\mathbf{AV} = \lambda \mathbf{V}$$

then $\mathbf{V}$ is called an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. Together, this eigenvalue λ and eigenvector $\mathbf{V}$ is called an eigenpair λ, $\mathbf{V}$.

The characteristic polynomial $p(\lambda) = (-1)^n \left(\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-2} \lambda^2 + c_{n-1} \lambda + c_n\right)$ can be factored in the form

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots \left(\lambda - \lambda_j\right)^{m_j} \ldots (\lambda - \lambda_{m_{k-1}})^{m_{k-1}} (\lambda - \lambda_{m_k})^{m_k}$$

where m_j is called the multiplicity of the eigenvalue λ_j. The sum of the multiplicities of all eigenvalues is n; that is,

$$n = m_1 + m_2 + \ldots + m_j + \ldots + m_{k-1} + m_k.$$

The next three results concern the existence of eigenvectors.

Theorem (Corresponding Eigenvectors). Suppose that $\mathbf{A}$ is and $n \times n$ square matrix.

(a) For each distinct eigenvalue λ there exists at least one eigenvector $\mathbf{V}$ corresponding to λ.

(b) If λ has multiplicity r, then there exist at most r linearly independent eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_r$ that correspond to λ.

Theorem (Linearly Independent Eigenvectors). Suppose that **A** is and $n \times n$ square matrix. If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct and $\lambda_1, \mathbf{V}_1; \lambda_{12}, \mathbf{V}_2; \ldots; \lambda_k, \mathbf{V}_k$ are the k eigenpairs, then $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k\}$ is a set of k linearly independent vectors.

Theorem (Complete Set of Eigenvectors). Suppose that **A** is and $n \times n$ square matrix. If the eigenvalues of **A** are all distinct, then there exist n nearly independent eigenvectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$.

Finding eigenpairs by hand computations is usually done in the following manner. The eigenvalue λ of multiplicity r is substituted into the equation

$$(\mathbf{A} - \lambda \mathbf{I}) \mathbf{V} = 0.$$

Then Gaussian elimination can be performed to obtain the row reduced echelon form, which will involve n-k equations in n unknowns, where $1 \leq k \leq r$. Hence there are k free variables to choose. The free variables can be selected in a judicious manner to produce k linearly independent solution vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k$ that correspond to λ.

```
NewtonRaphson[x0_, max_] :=
 Module[{},
  k = 0;
  p0 = N[x0];
  Print["p0 = ", PaddedForm[p0, {16, 16}],
   ",    f[p0] = ", NumberForm[f[p0], 16] ];
  p1 = p0;
  While[ k < max,
   p0 = p1;
   p1 = p0 - f[p0]/f'[p0];
   k = k + 1;
   Print["p"_k, " = ", PaddedForm[p1, {16, 16}],
    ",    f[", "p"_k, "] = ", NumberForm[f[p1], 16] ]; ];
  Print["  p   = ", NumberForm[p1, 16] ];
  Print[" Δp   = ±", Abs[p1 - p0] ];
  Print["f[p]  = ", NumberForm[f[p1], 16] ]; ]
```

Example 2. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Solution 2.

```
A = (1 2
     1 1);
I2 = IdentityMatrix[2];
M = A - λ I2 ;
p[λ_] = Det[M];
solset = Solve[p[λ] == 0, λ];
Print["  A  = ", MatrixForm[ A ] ];
Print["  M  = ", MatrixForm[ M ] ];
Print["p[λ] = Det[M] = ", p[λ]];
Print["Solve p[λ] = 0  get"];
Print[TableForm[solset]];
```

$A \; = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$M \; = \begin{pmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{pmatrix}$

$p[\lambda] = \mathrm{Det}[M] = -1 - 2\lambda + \lambda^2$

Solve $p[\lambda] = 0$ get

$\lambda \to 1 - \sqrt{2}$

$\lambda \to 1 + \sqrt{2}$

```
i = 1;
λi = 1 - √2 ;
B = {0, 0};
b = Partition[B, 1];
Needs["LinearAlgebra`MatrixManipulation`"];
M1 = AppendRows[M, b];
Print["The augmented matrix M is"];
Print["M"i, " = ", MatrixForm[M1]];
M1 = ReplaceAll[M, λ → λi];
M1 = AppendRows[M1, b];
Print["Substitute ", "λ"i, " = ", λi];
Print["The augmented matrix M is"];
Print["M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form is"];
Print[MatrixForm[ RowReduce[ M1] ]]
```

The augmented matrix M is

$$M_1 = \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 0 \end{pmatrix}$$

Substitute $\lambda_1 = 1 - \sqrt{2}$

The augmented matrix M is

$$M_1 = \begin{pmatrix} \sqrt{2} & 2 & 0 \\ 1 & \sqrt{2} & 0 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is equivalent to the linear system

$$x_1 + \sqrt{2}\, x_2 = 0$$

Set $x_2 = -1$ and solve for $x_1 = \sqrt{2}$, and get the eigenvector

$$V_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix};$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = (", λi, ")", MatrixForm[Vi], " = ",
  MatrixForm[λi Vi], " = ", ExpandAll[MatrixForm[λi Vi]]];
Print[ExpandAll[A.Vi] == ExpandAll[λi Vi]];
```

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$\lambda_1 = 1 - \sqrt{2}$

$V_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

Does $A\ V_1 = \lambda_1 V_1$?

$A\ V_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = \begin{pmatrix} -2 + \sqrt{2} \\ -1 + \sqrt{2} \end{pmatrix}$

$\lambda_1 V_1 = (1 - \sqrt{2}) \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\,(1 - \sqrt{2}) \\ -1 + \sqrt{2} \end{pmatrix} = \begin{pmatrix} -2 + \sqrt{2} \\ -1 + \sqrt{2} \end{pmatrix}$

True

```
i = 2;
λi = 1 + √2 ;
B = {0, 0};
b = Partition[B, 1];
Needs["LinearAlgebra`MatrixManipulation`"];
M1 = AppendRows[M, b];
Print["The augmented matrix M is"];
Print["M"i, " = ", MatrixForm[M1]];
M1 = ReplaceAll[M, λ → λi];
M1 = AppendRows[M1, b];
Print["Substitute ", "λ"i, " = ", λi];
Print["The augmented matrix M is"];
Print["M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form is"];
Print[MatrixForm[ RowReduce[ M1] ]]
```

The augmented matrix M is

$$M_2 = \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 0 \end{pmatrix}$$

Substitute $\lambda_2 = 1 + \sqrt{2}$

The augmented matrix M is

$$M_2 = \begin{pmatrix} -\sqrt{2} & 2 & 0 \\ 1 & -\sqrt{2} & 0 \end{pmatrix}$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is equivalent to the linear system

$$x_1 - \sqrt{2}\, x_2 = 0$$

Set $x_2 = 1$ and solve for $x_1 = \sqrt{2}$, and get the eigenvector

$$V_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix};$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = (", λi, ")", MatrixForm[Vi], " = ",
  MatrixForm[λi Vi], " = ", ExpandAll[MatrixForm[λi Vi]]];
Print[ExpandAll[A.Vi] == ExpandAll[λi Vi]];
```

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$\lambda_2 = 1 + \sqrt{2}$

$V_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

Does $A\ V_2 = \lambda_2 V_2$?

$A\ V_2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + \sqrt{2} \\ 1 + \sqrt{2} \end{pmatrix}$

$\lambda_2 V_2 = (1 + \sqrt{2}) \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\left(1 + \sqrt{2}\right) \\ 1 + \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 + \sqrt{2} \\ 1 + \sqrt{2} \end{pmatrix}$

True

Remark. Newton's method can be used to find the roots of the characteristic polynomial.

For the first eigenvalue.

```
f[x_] = p[x];
Print["p[x] = ", p[x]];
NewtonRaphson[-0.5, 4];
```

$p[x] = -1 - 2x + x^2$

$p_0 = -0.5000000000000000, \quad f[p_0] = 0.25$

$p_1 = -0.4166666666666667, \quad f[p_1] = 0.006944444444444503$

$p_2 = -0.4142156862745098, \quad f[p_2] = 6.007304882649223 \times 10^{-6}$

$p_3 = -0.4142135623746899, \quad f[p_3] = 4.511002682505705 \times 10^{-12}$

$p_4 = -0.4142135623730950, \quad f[p_4] = -2.775557561562891 \times 10^{-17}$

$p = -0.414213562373095$

$\Delta p = \pm 1.59489 \times 10^{-12}$

$f[p] = -2.775557561562891 \times 10^{-17}$

Which is an approximation to the eigenvalue

```
Print["λ"1, " = ", λ1, " = ", NumberForm[N[λ1], 17]];
```

$\lambda_1 = 1 - \sqrt{2} = -0.4142135623730951$

For the second eigenvalue.

```
f[x_] = p[x];
Print["p[x] = ", p[x]];
NewtonRaphson[1.5, 6];
```

$p[x] = -1 - 2x + x^2$

$p_0 = 1.5000000000000000, \quad f[p_0] = -1.75$

$p_1 = 3.2500000000000000, \quad f[p_1] = 3.0625$

$p_2 = 2.5694444444444450, \quad f[p_2] = 0.4631558641975317$

$p_3 = 2.4218903638151420, \quad f[p_3] = 0.02177220671035762$

$p_4 = 2.4142342859400730, \quad f[p_4] = 0.0000586155284292289$

$p_5 = 2.4142135625249320, \quad f[p_5] = 4.294600230991819 \times 10^{-10}$

$p_6 = 2.4142135623730950, \quad f[p_6] = 0.$

$p = 2.414213562373095$

$\Delta p = \pm 1.51837 \times 10^{-10}$

$f[p] = 0.$

Which is an approximation to the eigenvalue

```
Print["λ"2, " = ", λ2, " = ", NumberForm[N[λ2], 17]];
```

$\lambda_2 = 1 + \sqrt{2} = 2.414213562373095$

Free Variables

When the linear system is underdetermined, we needed to introduce free variables in the proper location. The following subroutine will rearrange the equations and introduce free variables in the location they are needed. Then all that is needed to do is find the row reduced echelon form a second time. This is done at the end of the next example.

```
FreeVariables[M0_] := Module[{c, i, k, L, M = M0, m, n, Z},
  n = Dimensions[M][[1]];
  m = Dimensions[M][[2]];
  L = {t, s, r, q, p, u, v, w, z, y, x};
  Z = Table[0, {m}];
  c = 1;
  For[i = 1, i ≤ n, i++,
   If[M[[i,i]] == 0,
     For[k = n, i < k, k--,
       M[[k]] = M[[k-1]] ]; ]; ];
  For[i = n, 1 ≤ i, i--,
   If[M[[i,i]] == 0,
     M[[i]] = Z;
     M[[i,i]] = 1;
     M[[i,m]] = L[[c]];
     c = c + 1; ]; ];
  Return[M]]
```

Example 3. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Solution 3.

Find the characteristic polynomial and the eigenvalues.

```
A = (  2  -1  1
      -1   2  1
       1  -1  2 );
n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];];
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 6 - 11\lambda + 6\lambda^2 - \lambda^3$

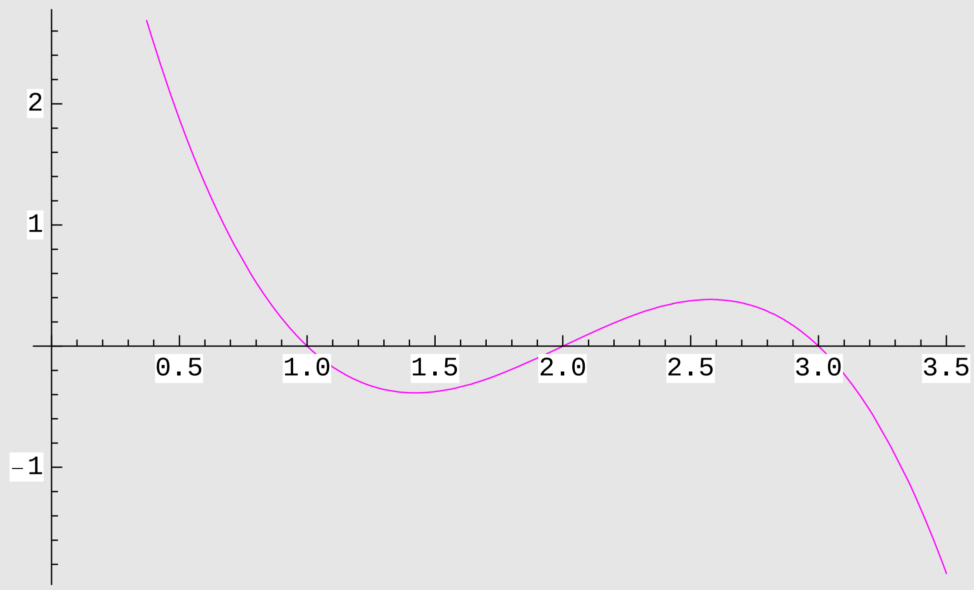
$p[\lambda] = -(-3+\lambda)(-2+\lambda)(-1+\lambda)$

To find the eigenvalues of the matrix A

Solve $6 - 11\lambda + 6\lambda^2 - \lambda^3 == 0$

Let us plot $p[\lambda] = 6 - 11\lambda + 6\lambda^2 - \lambda^3$ and see where the roots are located

```
Needs["Graphics`Colors`"]
Plot[p[λ], {λ, 0, 3.5`}, PlotStyle → Magenta]
Print["p[λ] = ", p[λ]];
```



p[λ] = $6 - 11\lambda + 6\lambda^2 - \lambda^3$

Although this example has been "cooked up" so that the values are simple, we should be aware that a root finding method could be employed to find the eigenvalues. For illustration, we can use the Newton-Raphson method.

```
f[x_] = p[x];
```

```
NewtonRaphson[1.2, 6];
```

p_0 = 1.2000000000000000, $f[p_0]$ = −0.2879999999999985

p_1 = 0.8869565217391330, $f[p_1]$ = 0.2658680036163332

p_2 = 0.9848245415522550, $f[p_2]$ = 0.03104529533799827

p_3 = 0.9996663676866180, $f[p_3]$ = 0.0006675985954629482

p_4 = 0.9999998331640960, $f[p_4]$ = $3.336718910063041 \times 10^{-7}$

p_5 = 0.9999999999999580, $f[p_5]$ = $8.26005930321117 \times 10^{-14}$

p_6 = 1.0000000000000000, $f[p_6]$ = 0.

p = 1.

$\triangle p$ = $\pm 4.13003 \times 10^{-14}$

f[p] = 0.

```
NewtonRaphson[2.2, 4];
```

p_0 = 2.2000000000000000, $f[p_0]$ = 0.1920000000000002

p_1 = 1.9818181818181810, $f[p_1]$ = −0.01817580766341464

p_2 = 2.0000120329703420, $f[p_2]$ = 0.000012032970342801 26

p_3 = 1.9999999999999940, $f[p_3]$ = $-7.105427357601002 \times 10^{-15}$

p_4 = 2.0000000000000010, $f[p_4]$ = 0.

p = 2.000000000000001

$\triangle p$ = $\pm 7.10543 \times 10^{-15}$

f[p] = 0.

```
NewtonRaphson[3.2, 5];
```

p_0 = 3.2000000000000000, $f[p_0]$ = −0.5279999999999987

p_1 = 3.0409638554216870, $f[p_1]$ = −0.0870305620799421

p_2 = 3.0022976499455250, $f[p_2]$ = −0.004611149606606801

p_3 = 3.0000078765674940, $f[p_3]$ = −0.00001575332110448358

p_4 = 3.0000000000930620, $f[p_4]$ = $-1.861266696323582 \times 10^{-10}$

p_5 = 2.9999999999999980, $f[p_5]$ = $7.105427357601002 \times 10^{-15}$

p = 2.999999999999998

$\triangle p$ = $\pm 9.30633 \times 10^{-11}$

f[p] = $7.105427357601002 \times 10^{-15}$

Since we have solved for roots in previous modules, we will concentrate our effort on solving for the eigenvectors.

First, we shall automate the procedure for finding the roots of the characteristic polynomial, which is one way to find the eigenvalues.

```
A = ( 2  -1  1
     -1   2  1
      1  -1  2 );
n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 6 - 11\lambda + 6\lambda^2 - \lambda^3$

$p[\lambda] = -(-3+\lambda)(-2+\lambda)(-1+\lambda)$

To find the eigenvalues of the matrix A

Solve $6 - 11\lambda + 6\lambda^2 - \lambda^3 == 0$

Get

$\lambda_1 = 1 = 1.$

$\lambda_2 = 2 = 2.$

$\lambda_3 = 3 = 3.$

Investigate the eigen-pair λ_1, V_1

```
i = 1;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_1 = 1$

Solve the equation A_1 X = 0

$$A_1\, X = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_1 = [A,B] is

$$M_1 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_1 is

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvector is in the last column

$$V_1 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_1 = 1$

$$V_1 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

Does $A\ V_1 = \lambda_1 V_1$?

$$A\ V_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

$$\lambda_1 V_1 = 1 \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

True

Investigate the eigen-pair λ_2, V_2

```
i = 2;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue    ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_2 = 2$

Solve the equation A_2 X = 0

$$A_2\ X = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_2 = [A,B] is

$$M_2 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_2 is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_2 = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_2 = 2$

$$V_2 = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$$

Does $A\ V_2 = \lambda_2 V_2$?

$$A\ V_2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ 2t \end{pmatrix}$$

$$\lambda_2 V_2 = 2 \begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ 2t \end{pmatrix}$$

True

Investigate the eigen-pair λ_3, V_3

```
i = 3;
Aᵢ = ReplaceAll[M, λ → λᵢ];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mᵢ = AppendRows[Aᵢ, Partition[B, 1]];
RMᵢ = RowReduce[ Mᵢ];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue    ", "λ"ᵢ, " = ", λᵢ];
Print["Solve the equation ", "A"ᵢ , " X = 0"];
Print["A"ᵢ , " X = ",
  MatrixForm[Aᵢ] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"ᵢ, " = [A,B] is"];
Print["  ", "M"ᵢ, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"ᵢ, " is"];
Print["     ", MatrixForm[ RMᵢ ]];
```

For the eigenvalue $\lambda_3 = 3$

Solve the equation A_3 X = 0

$$A_3\ X = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_3 = [A,B] is

$$M_3 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_3 is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_3 = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"ᵢ, " = ", λᵢ];
Print["V"ᵢ, " = ", MatrixForm[Vᵢ]];
Print["Does  ", "A ", "V"ᵢ, " = ", "λ"ᵢ, "V"ᵢ, " ?"];
Print["A ", "V"ᵢ, " = ", MatrixForm[A],
  MatrixForm[Vᵢ], " = ", MatrixForm[A.Vᵢ]];
Print["λ"ᵢ, "V"ᵢ, " = ", λᵢ, MatrixForm[Vᵢ], " = ", MatrixForm[λᵢ Vᵢ]];
Print[ExpandAll[A.Vᵢ == λᵢ Vᵢ]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_3 = 3$

$$V_3 = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

Does $A\ V_3 = \lambda_3 V_3$?

$$A\ V_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 3\,t \\ 0 \\ 3\,t \end{pmatrix}$$

$$\lambda_3 V_3 = 3 \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 3\,t \\ 0 \\ 3\,t \end{pmatrix}$$

True

The three eigen-pairs are:

```
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ 3, i++,
  Print["λ"ᵢ, " = ", λᵢ, ",   ", "V"ᵢ, " = ", MatrixForm[Vᵢ]]; ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda_1 = 1, \quad V_1 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2, \quad V_2 = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$$

$$\lambda_3 = 3, \quad V_3 = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

We can compare this with the results obtained using *Mathemati-*

ca's **Eigensystem** procedure.

```
sol = Eigensystem[A];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"i, " = ", sol[[1,i]], ",   ", "V"i, " = ", MatrixForm[sol[[2,i]]]]; ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda_1 = 3, \quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2, \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 1, \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Example 4. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Solution 4.

Find the characteristic polynomial and the eigenvalues.

```
A = ( 2  -1  1
     -1   2 -1
      1  -1  2 );
n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A - \lambda I_n|$

$p[\lambda] = 4 - 9\lambda + 6\lambda^2 - \lambda^3$

$p[\lambda] = -(-4+\lambda)(-1+\lambda)^2$

To find the eigenvalues of the matrix A

Solve $4 - 9\lambda + 6\lambda^2 - \lambda^3 == 0$

Get

$\lambda_1 = 1 = 1.$

$\lambda_2 = 1 = 1.$

$\lambda_3 = 4 = 4.$

Investigate the eigen-pairs λ_1, V_1 and λ_2, V_2.

```
i = 1;
A_i = ReplaceAll[M, λ → λ_i];
vars = {x_1 , x_2 , x_3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
M_i = AppendRows[A_i, Partition[B, 1]];
RM_i = RowReduce[ M_i];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"_i, " = ", λ_i];
Print["Solve the equation ", "A"_i , " X = 0"];
Print["A"_i , " X = ",
  MatrixForm[A_i] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"_i, " = [A,B] is"];
Print["  ", "M"_i, " = ", MatrixForm[M_1]];
Print["The row reduced echelon form for ", "M"_i, " is"];
Print["     ", MatrixForm[ RM_i ]];
```

For the eigenvalue $\lambda_1 = 1$

Solve the equation A_1 X = 0

$$A_1\,X = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_1 = [A,B] is

$$M_1 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_1 is

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & s-t \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_1 = \begin{pmatrix} s-t \\ s \\ t \end{pmatrix}$$

The eigenvalue is repeated, and there are two linearly independent eigenvectors.

Investigate the eigen-pairs λ_1, V_1 and λ_2, V_2.

```
W = Vi;
Print["W = ", MatrixForm[W]];
```

$$W = \begin{pmatrix} s-t \\ s \\ t \end{pmatrix}$$

For V_1, set $s=0$ in **W** and get

```
λ1 = λ1;
Vi = ReplaceAll[W, {s → 0}];
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_1 = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_1 = 1$

$$V_1 = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

Does $A\ V_1 = \lambda_1 V_1$?

$$A\ V_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$\lambda_1 V_1 = 1 \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

True

For V_2, set $t=0$ in **W** and get

```
i = 2;
λi = λ1;
Vi = ReplaceAll[W, {t → 0}];
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_2 = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"_i, " = ", λ_i];
Print["V"_i, " = ", MatrixForm[V_i]];
Print["Does  ", "A ", "V"_i, " = ", "λ"_i, "V"_i, " ?"];
Print["A ", "V"_i, " = ", MatrixForm[A],
  MatrixForm[V_i], " = ", MatrixForm[A.V_i]];
Print["λ"_i, "V"_i, " = ", λ_i, MatrixForm[V_i], " = ", MatrixForm[λ_i V_i]];
Print[ExpandAll[A.V_i == λ_i V_i]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_2 = 1$

$$V_2 = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

Does $A\ V_2 = \lambda_2 V_2$?

$$A\ V_2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

$$\lambda_2 V_2 = 1 \begin{pmatrix} s \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

True

Investigate the eigen-pair λ_3, V_3

```
i = 3;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_3 = 4$

Solve the equation A_3 X = 0

$$A_3\ X = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_3 = [A,B] is

$$M_3 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_3 is

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_3 = \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\lambda_3 = 4$

$$V_3 = \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$$

Does A V_3 = $\lambda_3 V_3$?

$$A\ V_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = \begin{pmatrix} 4\,t \\ -4\,t \\ 4\,t \end{pmatrix}$$

$$\lambda_3 V_3 = 4 \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = \begin{pmatrix} 4\,t \\ -4\,t \\ 4\,t \end{pmatrix}$$

True

It was good fortune that the three eigenvectors are linearly independent.

```
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ 3, i++,
  Print["λ"i, " = ", λi, ",   ", "V"i, " = ", MatrixForm[Vi]]; ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda_1 = 1, \quad V_1 = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$\lambda_2 = 1, \quad V_2 = \begin{pmatrix} s \\ s \\ 0 \end{pmatrix}$$

$$\lambda_3 = 4, \quad V_3 = \begin{pmatrix} t \\ -t \\ t \end{pmatrix}$$

We can compare this with the results obtained using Mathematicas **Eigensystem** procedure.

```
sol = Eigensystem[A];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"_i, " = ", sol[[1,i]], ",   ", "V"_i, " = ", MatrixForm[sol[[2,i]]]]; ];
```

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda_1 = 4, \quad V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \quad V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 1, \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Example 5. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$.

Solution 5.

Find the characteristic polynomial and the eigenvalues.

```
A = ( 1   0   3
      1  -1  -3
     -1   2  -1 );
n = Length[A];
Iₙ = IdentityMatrix[n];
M = A - λ Iₙ;
p[λ_] = Det[A - λ Iₙ];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λᵢ = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"ₙ, " = ", MatrixForm[A], " - λ", MatrixForm[Iₙ]];
Print["A - λ", "I"ₙ, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ Iₙ|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"ᵢ, " = ", λᵢ, " = ", Chop[N[λᵢ]]] ];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_3 = \begin{pmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & -3 \\ -1 & 2 & -1-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 10 - 8\lambda - \lambda^2 - \lambda^3$

$p[\lambda] = -(-1+\lambda)\left(10 + 2\lambda + \lambda^2\right)$

To find the eigenvalues of the matrix A

Solve $10 - 8\lambda - \lambda^2 - \lambda^3 == 0$

Get

$\lambda_1 = -1 - 3 i = -1. - 3. i$

$\lambda_2 = -1 + 3 i = -1. + 3. i$

$\lambda_3 = 1 = 1.$

Investigate the eigen-pair λ_1, V_1

```
i = 1;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_1 = -1 - 3\,i$

Solve the equation A_1 X = 0

$$A_1\ X = \begin{pmatrix} 2+3\,i & 0 & 3 \\ 1 & 3\,i & -3 \\ -1 & 2 & 3\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_1 = [A,B] is

$$M_1 = \begin{pmatrix} 2+3\,i & 0 & 3 & 0 \\ 1 & 3\,i & -3 & 0 \\ -1 & 2 & 3\,i & 0 \end{pmatrix}$$

The row reduced echelon form for M_1 is

$$\begin{pmatrix} 1 & 0 & \frac{6}{13} - \frac{9\,i}{13} & 0 \\ 0 & 1 & \frac{3}{13} + \frac{15\,i}{13} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & \frac{6}{13} - \frac{9\,i}{13} & 0 \\ 0 & 1 & \frac{3}{13} + \frac{15\,i}{13} & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & \left(-\frac{6}{13} + \frac{9\,i}{13}\right) t \\ 0 & 1 & 0 & \left(-\frac{3}{13} - \frac{15\,i}{13}\right) t \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_1 = \begin{pmatrix} \left(-\frac{6}{13} + \frac{9\,i}{13}\right) t \\ \left(-\frac{3}{13} - \frac{15\,i}{13}\right) t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 13t.

```
Vi = 13 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_1 = \begin{pmatrix} (-6 + 9\,i)\,t \\ (-3 - 15\,i)\,t \\ 13\,t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"ᵢ, " = ", λᵢ];
Print["V"ᵢ, " = ", MatrixForm[Vᵢ]];
Print["Does  ", "A ", "V"ᵢ, " = ", "λ"ᵢ, "V"ᵢ, " ?"];
Print["A ", "V"ᵢ, " = ", MatrixForm[A],
  MatrixForm[Vᵢ], " = ", MatrixForm[A.Vᵢ]];
Print["λ"ᵢ, "V"ᵢ, " = (", λᵢ, ")", MatrixForm[Vᵢ],
  " = ", MatrixForm[λᵢ Vᵢ]];
Print[ExpandAll[A.Vᵢ == λᵢ Vᵢ]];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$\lambda_1 = -1 - 3\,i$

$$V_1 = \begin{pmatrix} (-6 + 9\,i)\,t \\ (-3 - 15\,i)\,t \\ 13\,t \end{pmatrix}$$

Does A V_1 = $\lambda_1 V_1$?

$$A\ V_1 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} (-6 + 9\,i)\,t \\ (-3 - 15\,i)\,t \\ 13\,t \end{pmatrix} = \begin{pmatrix} (33 + 9\,i)\,t \\ (-42 + 24\,i)\,t \\ (-13 - 39\,i)\,t \end{pmatrix}$$

$$\lambda_1 V_1 = (-1 - 3\,i) \begin{pmatrix} (-6 + 9\,i)\,t \\ (-3 - 15\,i)\,t \\ 13\,t \end{pmatrix} = \begin{pmatrix} (33 + 9\,i)\,t \\ (-42 + 24\,i)\,t \\ (-13 - 39\,i)\,t \end{pmatrix}$$

True

Investigate the eigen-pair λ_2, V_2

```
i = 2;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue    ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_2 = -1+3\,i$

Solve the equation A_2 X = 0

$$A_2\ X = \begin{pmatrix} 2-3\,i & 0 & 3 \\ 1 & -3\,i & -3 \\ -1 & 2 & -3\,i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_2 = [A,B] is

$$M_2 = \begin{pmatrix} 2+3\,i & 0 & 3 & 0 \\ 1 & 3\,i & -3 & 0 \\ -1 & 2 & 3\,i & 0 \end{pmatrix}$$

The row reduced echelon form for M_2 is

$$\begin{pmatrix} 1 & 0 & \frac{6}{13}+\frac{9\,i}{13} & 0 \\ 0 & 1 & \frac{3}{13}-\frac{15\,i}{13} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & \frac{6}{13}+\frac{9i}{13} & 0 \\ 0 & 1 & \frac{3}{13}-\frac{15i}{13} & 0 \\ 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & \left(-\frac{6}{13}-\frac{9i}{13}\right)t \\ 0 & 1 & 0 & \left(-\frac{3}{13}+\frac{15i}{13}\right)t \\ 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_2 = \begin{pmatrix} \left(-\frac{6}{13}-\frac{9i}{13}\right)t \\ \left(-\frac{3}{13}+\frac{15i}{13}\right)t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 13t.

```
Vi = 13 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_2 = \begin{pmatrix} (-6-9i)t \\ (-3+15i)t \\ 13t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = (", λi, ")", MatrixForm[Vi],
  " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\lambda_2 = -1 + 3\,i$$

$$V_2 = \begin{pmatrix} (-6 - 9\,i)\,t \\ (-3 + 15\,i)\,t \\ 13\,t \end{pmatrix}$$

Does $A\ V_2 = \lambda_2 V_2$?

$$A\ V_2 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} (-6 - 9\,i)\,t \\ (-3 + 15\,i)\,t \\ 13\,t \end{pmatrix} = \begin{pmatrix} (33 - 9\,i)\,t \\ (-42 - 24\,i)\,t \\ (-13 + 39\,i)\,t \end{pmatrix}$$

$$\lambda_2 V_2 = (-1 + 3\,i) \begin{pmatrix} (-6 - 9\,i)\,t \\ (-3 + 15\,i)\,t \\ 13\,t \end{pmatrix} = \begin{pmatrix} (33 - 9\,i)\,t \\ (-42 - 24\,i)\,t \\ (-13 + 39\,i)\,t \end{pmatrix}$$

True

Investigate the eigen-pair λ_3, V_3

```
i = 3;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3};
B = {0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_3 = 1$

Solve the equation A_3 X = 0

$$A_3\ X = \begin{pmatrix} 0 & 0 & 3 \\ 1 & -2 & -3 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_3 = [A,B] is

$$M_3 = \begin{pmatrix} 2+3\,i & 0 & 3 & 0 \\ 1 & 3\,i & -3 & 0 \\ -1 & 2 & 3\,i & 0 \end{pmatrix}$$

The row reduced echelon form for M_3 is

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 2t \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvector is in the last column

$$V_3 = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$\lambda_3 = 1$

$$V_3 = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

Does $A\ V_3 = \lambda_3 V_3$?

$$A\ V_3 = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

$$\lambda_3 V_3 = 1 \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

True

The three eigen-pairs are:

```
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ 3, i++,
  Print["λ"i, " = ", λi, ",   ", "V"i, " = ", MatrixForm[Vi]]; ];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\lambda_1 = -1 - 3i, \quad V_1 = \begin{pmatrix} (-6 + 9i)\,t \\ (-3 - 15i)\,t \\ 13\,t \end{pmatrix}$$

$$\lambda_2 = -1 + 3i, \quad V_2 = \begin{pmatrix} (-6 - 9i)\,t \\ (-3 + 15i)\,t \\ 13\,t \end{pmatrix}$$

$$\lambda_3 = 1, \quad V_3 = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix}$$

We can compare this with the results obtained using Mathematicas

Eigensystem procedure.

```
sol = Eigensystem[A];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"_i, " = ", sol[[1,i]], ",   ", "V"_i, " = ", MatrixForm[sol[[2,i]]]]; ];
```

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & -3 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\lambda_1 = -1 + 3\,i, \quad V_1 = \begin{pmatrix} -\frac{6}{13} - \frac{9\,i}{13} \\ -\frac{3}{13} + \frac{15\,i}{13} \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 - 3\,i, \quad V_2 = \begin{pmatrix} -\frac{6}{13} + \frac{9\,i}{13} \\ -\frac{3}{13} - \frac{15\,i}{13} \\ 1 \end{pmatrix}$$

$$\lambda_3 = 1, \quad V_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Example 6. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Do this by constructing the characteristic polynomial and finding its roots, and compare with *Mathematica*'s **Eigenvalues** procedure.

Solution 6.

Find the characteristic polynomial and the eigenvalues.

```
A = (1  -1  0   0
     1   2  0   0
     0   1  1  -3
     0   0  1   2);

n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$A - \lambda I_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_4 = \begin{pmatrix} 1-\lambda & -1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 1 & 1-\lambda & -3 \\ 0 & 0 & 1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 15 - 24\lambda + 17\lambda^2 - 6\lambda^3 + \lambda^4$

$p[\lambda] = (3 - 3\lambda + \lambda^2)(5 - 3\lambda + \lambda^2)$

To find the eigenvalues of the matrix A

Solve $15 - 24\lambda + 17\lambda^2 - 6\lambda^3 + \lambda^4 == 0$

Get

$$\lambda_1 = \frac{1}{2}\left(3 - i\sqrt{3}\right) = 1.5 - 0.866025\,i$$

$$\lambda_2 = \frac{1}{2}\left(3 + i\sqrt{3}\right) = 1.5 + 0.866025\,i$$

$$\lambda_3 = \frac{1}{2}\left(3 - i\sqrt{11}\right) = 1.5 - 1.65831\,i$$

$$\lambda_4 = \frac{1}{2}\left(3 + i\sqrt{11}\right) = 1.5 + 1.65831\,i$$

Investigate the eigen-pair λ_1, V_1

```
i = 1;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4};
B = {0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_1 = \frac{1}{2}\left(3 - i\sqrt{3}\right)$

Solve the equation A_1 X = 0

A_1 X =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -1 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -3 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_1 = [A,B] is

M_1 =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -1 & 0 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -3 & 0 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 \end{pmatrix}$$

The row reduced echelon form for M_1 is

$$\begin{pmatrix} 1 & 0 & 0 & 1+i\sqrt{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1+i\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 1+i\sqrt{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1+i\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -i\left(-i+\sqrt{3}\right)t \\ 0 & 1 & 0 & 0 & 2t \\ 0 & 0 & 1 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right)t \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_1 = \begin{pmatrix} -i\left(-i+\sqrt{3}\right)t \\ 2t \\ -\frac{1}{2}i\left(-i+\sqrt{3}\right)t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 2t.

```
Vi = 2 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_1 = \begin{pmatrix} -2\,i\left(-i+\sqrt{3}\right)t \\ 4\,t \\ -i\left(-i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = (", λi, ")", MatrixForm[Vi],
  " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = \frac{1}{2}\left(3 - i\sqrt{3}\right)$$

$$V_1 = \begin{pmatrix} -2\,i\left(-i+\sqrt{3}\right)t \\ 4\,t \\ -i\left(-i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix}$$

Does A V_1 = $\lambda_1 V_1$?

$$A\,V_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -2\,i\left(-i+\sqrt{3}\right)t \\ 4\,t \\ -i\left(-i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix} = \begin{pmatrix} -4\,t-2\,i\left(-i+\sqrt{3}\right)t \\ 8\,t-2\,i\left(-i+\sqrt{3}\right)t \\ -2\,t-i\left(-i+\sqrt{3}\right)t \\ 4\,t-i\left(-i+\sqrt{3}\right)t \end{pmatrix}$$

$$\lambda_1 V_1 = \left(\frac{1}{2}\left(3-i\sqrt{3}\right)\right)\begin{pmatrix} -2\,i\left(-i+\sqrt{3}\right)t \\ 4\,t \\ -i\left(-i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix} = \begin{pmatrix} -i\left(3-i\sqrt{3}\right)\left(-i+\sqrt{3}\right)t \\ 2\left(3-i\sqrt{3}\right)t \\ -\frac{1}{2}\,i\left(3-i\sqrt{3}\right)\left(-i+\sqrt{3}\right)t \\ \left(3-i\sqrt{3}\right)t \end{pmatrix}$$

True

Investigate the eigen-pair λ_2, V_2

```
i = 2;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4};
B = {0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_2 = \frac{1}{2}\left(3 + i\sqrt{3}\right)$

Solve the equation $A_2\ X = 0$

$A_2\ X =$

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3-i\sqrt{3}\right) & -1 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3-i\sqrt{3}\right) & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3-i\sqrt{3}\right) & -3 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3-i\sqrt{3}\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix $M_2 = [A,B]$ is

$M_2 =$

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -1 & 0 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -3 & 0 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 \end{pmatrix}$$

The row reduced echelon form for M_2 is

$$\begin{pmatrix} 1 & 0 & 0 & 1 - i\sqrt{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1 - i\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 1 - i\sqrt{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1 - i\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & i\left(i + \sqrt{3}\right) t \\ 0 & 1 & 0 & 0 & 2\, t \\ 0 & 0 & 1 & 0 & \frac{1}{2}\, i\left(i + \sqrt{3}\right) t \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_2 = \begin{pmatrix} i\left(i + \sqrt{3}\right) t \\ 2\, t \\ \frac{1}{2}\, i\left(i + \sqrt{3}\right) t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 2t.

```
Vi = 2 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_2 = \begin{pmatrix} 2\,i\left(i+\sqrt{3}\right)t \\ 4\,t \\ i\left(i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"ᵢ, " = ", λᵢ];
Print["V"ᵢ, " = ", MatrixForm[Vᵢ]];
Print["Does  ", "A ", "V"ᵢ, " = ", "λ"ᵢ, "V"ᵢ, " ?"];
Print["A ", "V"ᵢ, " = ", MatrixForm[A],
  MatrixForm[Vᵢ], " = ", MatrixForm[A.Vᵢ]];
Print["λ"ᵢ, "V"ᵢ, " = (", λᵢ, ")", MatrixForm[Vᵢ],
  " = ", MatrixForm[λᵢ Vᵢ]];
Print[ExpandAll[A.Vᵢ == λᵢ Vᵢ]];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}\left(3 + i\sqrt{3}\right)$$

$$V_2 = \begin{pmatrix} 2\,i\left(i+\sqrt{3}\right)t \\ 4\,t \\ i\left(i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix}$$

Does A V_2 = $\lambda_2 V_2$?

$$A\ V_2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2\,i\left(i+\sqrt{3}\right)t \\ 4\,t \\ i\left(i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix} = \begin{pmatrix} -4\,t + 2\,i\left(i+\sqrt{3}\right)t \\ 8\,t + 2\,i\left(i+\sqrt{3}\right)t \\ -2\,t + i\left(i+\sqrt{3}\right)t \\ 4\,t + i\left(i+\sqrt{3}\right)t \end{pmatrix}$$

$$\lambda_2 V_2 = \left(\frac{1}{2}\left(3 + i\sqrt{3}\right)\right) \begin{pmatrix} 2\,i\left(i+\sqrt{3}\right)t \\ 4\,t \\ i\left(i+\sqrt{3}\right)t \\ 2\,t \end{pmatrix} = \begin{pmatrix} i\left(3+i\sqrt{3}\right)\left(i+\sqrt{3}\right)t \\ 2\left(3+i\sqrt{3}\right)t \\ \frac{1}{2}\,i\left(3+i\sqrt{3}\right)\left(i+\sqrt{3}\right)t \\ \left(3+i\sqrt{3}\right)t \end{pmatrix}$$

True

Investigate the eigen-pair λ_3, V_3

```
i = 3;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4};
B = {0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_3 = \frac{1}{2}\left(3 - i\sqrt{11}\right)$

Solve the equation A_3 X = 0

A_3 X =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{11}\right) & -1 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{11}\right) & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{11}\right) & -3 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{11}\right) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_3 = [A,B] is

M_3 =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -1 & 0 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -3 & 0 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 \end{pmatrix}$$

The row reduced echelon form for M_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1+i\sqrt{11}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1+i\sqrt{11}\right) & 0 \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2}i\left(-i+\sqrt{11}\right)t \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}i\left(-i+\sqrt{11}\right)t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 2t.

```
Vi = 2 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ -i\left(-i+\sqrt{11}\right)t \\ 2t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{2}\left(3 - i\sqrt{11}\right)$$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ -i\left(-i+\sqrt{11}\right)t \\ 2t \end{pmatrix}$$

Does A V_3 = $\lambda_3 V_3$?

$$A\ V_3 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -i\left(-i+\sqrt{11}\right)t \\ 2t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6t - i\left(-i+\sqrt{11}\right)t \\ 4t - i\left(-i+\sqrt{11}\right)t \end{pmatrix}$$

$$\lambda_3 V_3 = \frac{1}{2}\left(3 - i\sqrt{11}\right) \begin{pmatrix} 0 \\ 0 \\ -i\left(-i+\sqrt{11}\right)t \\ 2t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}i\left(3-i\sqrt{11}\right)\left(-i+\sqrt{11}\right)t \\ \left(3-i\sqrt{11}\right)t \end{pmatrix}$$

True

Investigate the eigen-pair λ_4, V_4

```
i = 4;
A_i = ReplaceAll[M, λ → λ_i];
vars = {x_1 , x_2 , x_3, x_4};
B = {0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
M_i = AppendRows[A_i, Partition[B, 1]];
RM_i = RowReduce[ M_i];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"_i, " = ", λ_i];
Print["Solve the equation ", "A"_i , " X = 0"];
Print["A"_i , " X = ",
  MatrixForm[A_i] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"_i, " = [A,B] is"];
Print["  ", "M"_i, " = ", MatrixForm[M_1]];
Print["The row reduced echelon form for ", "M"_i, " is"];
Print["     ", MatrixForm[ RM_i ]];
```

For the eigenvalue $\lambda_4 = \frac{1}{2}\left(3 + i\sqrt{11}\right)$

Solve the equation A_4 X = 0

A_4 X =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3-i\sqrt{11}\right) & -1 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3-i\sqrt{11}\right) & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3-i\sqrt{11}\right) & -3 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3-i\sqrt{11}\right) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_4 = [A,B] is

M_4 =

$$\begin{pmatrix} 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -1 & 0 & 0 & 0 \\ 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 1 & 1+\frac{1}{2}\left(-3+i\sqrt{3}\right) & -3 & 0 \\ 0 & 0 & 1 & 2+\frac{1}{2}\left(-3+i\sqrt{3}\right) & 0 \end{pmatrix}$$

The row reduced echelon form for M_4 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1 - i\sqrt{11}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2}\left(1 - i\sqrt{11}\right) & 0 \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} i \left(i + \sqrt{11}\right) t \\ 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_4 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} i \left(i + \sqrt{11}\right) t \\ t \end{pmatrix}$$

In this case the eigenvector will have a nicer appearance if we replace t with 2t.

```
Vi = 2 Vi;
Print["V"i, " = ", MatrixForm[Vi]];
```

$$V_4 = \begin{pmatrix} 0 \\ 0 \\ i \left(i + \sqrt{11}\right) t \\ 2\, t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_4 = \frac{1}{2}\left(3 + i\sqrt{11}\right)$$

$$V_4 = \begin{pmatrix} 0 \\ 0 \\ i\left(i + \sqrt{11}\right) t \\ 2\,t \end{pmatrix}$$

Does $A\ V_4 = \lambda_4 V_4$?

$$A\ V_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ i\left(i + \sqrt{11}\right) t \\ 2\,t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6\,t + i\left(i + \sqrt{11}\right) t \\ 4\,t + i\left(i + \sqrt{11}\right) t \end{pmatrix}$$

$$\lambda_4 V_4 = \frac{1}{2}\left(3 + i\sqrt{11}\right) \begin{pmatrix} 0 \\ 0 \\ i\left(i + \sqrt{11}\right) t \\ 2\,t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\, i\left(3 + i\sqrt{11}\right)\left(i + \sqrt{11}\right) t \\ \left(3 + i\sqrt{11}\right) t \end{pmatrix}$$

True

The four eigen-pairs are:

```
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ 4, i++,
  Print["λ"i, " = ", λi, ",   ", "V"i, " = ", MatrixForm[Vi]]; ];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = \frac{1}{2}\left(3 - i\sqrt{3}\right), \quad V_1 = \begin{pmatrix} -2\,i\left(-i + \sqrt{3}\right) t \\ 4\,t \\ -i\left(-i + \sqrt{3}\right) t \\ 2\,t \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}\left(3 + i\sqrt{3}\right), \quad V_2 = \begin{pmatrix} 2\,i\left(i + \sqrt{3}\right) t \\ 4\,t \\ i\left(i + \sqrt{3}\right) t \\ 2\,t \end{pmatrix}$$

$$\lambda_3 = \frac{1}{2}\left(3 - i\sqrt{11}\right), \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ -i\left(-i + \sqrt{11}\right) t \\ 2\,t \end{pmatrix}$$

$$\lambda_4 = \frac{1}{2}\left(3 + i\sqrt{11}\right), \quad V_4 = \begin{pmatrix} 0 \\ 0 \\ i\left(i + \sqrt{11}\right) t \\ 2\,t \end{pmatrix}$$

We can compare this with the results obtained using Mathematicas **Eigensystem** procedure.

```
sol = Eigensystem[A];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"_i, " = ", sol[[1,i]], ",   ", "V"_i, " = ", MatrixForm[sol[[2,i]]]; ];
```

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = \frac{1}{2}\left(3 + i\sqrt{11}\right), \quad V_1 = \begin{pmatrix} 0 \\ 0 \\ -2 + \frac{1}{2}\left(3 + i\sqrt{11}\right) \\ 1 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}\left(3 - i\sqrt{11}\right), \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ -2 + \frac{1}{2}\left(3 - i\sqrt{11}\right) \\ 1 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{2}\left(3 + i\sqrt{3}\right), \quad V_3 = \begin{pmatrix} -1 + i\sqrt{3} \\ 2 \\ -2 + \frac{1}{2}\left(3 + i\sqrt{3}\right) \\ 1 \end{pmatrix}$$

$$\lambda_4 = \frac{1}{2}\left(3 - i\sqrt{3}\right), \quad V_4 = \begin{pmatrix} -1 - i\sqrt{3} \\ 2 \\ -2 + \frac{1}{2}\left(3 - i\sqrt{3}\right) \\ 1 \end{pmatrix}$$

Example 7. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Solution 7.

Find the characteristic polynomial and the eigenvalues.

```
A = {{2, 1, 0, 0, 0}, {1, 2, 1, 0, 0}, {0, 1, 2, 1, 0}, {0, 0, 1, 2, 1}, {0, 0, 0, 1, 2}};

n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$A - \lambda I_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_5 = \begin{pmatrix} 2-\lambda & 1 & 0 & 0 & 0 \\ 1 & 2-\lambda & 1 & 0 & 0 \\ 0 & 1 & 2-\lambda & 1 & 0 \\ 0 & 0 & 1 & 2-\lambda & 1 \\ 0 & 0 & 0 & 1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 6 - 35\lambda + 56\lambda^2 - 36\lambda^3 + 10\lambda^4 - \lambda^5$

$p[\lambda] = -(-3+\lambda)(-2+\lambda)(-1+\lambda)\left(1 - 4\lambda + \lambda^2\right)$

To find the eigenvalues of the matrix A

Solve $6 - 35\lambda + 56\lambda^2 - 36\lambda^3 + 10\lambda^4 - \lambda^5 == 0$

Get

$\lambda_1 = 1 = 1.$

$\lambda_2 = 2 = 2.$

$\lambda_3 = 3 = 3.$

$\lambda_4 = 2 - \sqrt{3} = 0.267949$

$\lambda_5 = 2 + \sqrt{3} = 3.73205$

Investigate the eigen-pair λ_1, V_1

```
i = 1;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4 , x5};
B = {0, 0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_1 = 1$

Solve the equation A_1 X = 0

$$A_1\,X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_1 = [A,B] is

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_1 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -t \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -t \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_1 = \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$\lambda_1 = 1$

$$V_1 = \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix}$$

Does $A\ V_1 = \lambda_1 V_1$?

$$A\ V_1 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix}$$

$$\lambda_1 V_1 = 1 \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix}$$

True

Investigate the eigen-pair λ_2, V_2

```
i = 2;
A_i = ReplaceAll[M, λ → λ_i];
vars = {x_1 , x_2 , x_3, x_4 , x_5};
B = {0, 0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
M_i = AppendRows[A_i, Partition[B, 1]];
RM_i = RowReduce[ M_i];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"_i, " = ", λ_i];
Print["Solve the equation ", "A"_i , " X = 0"];
Print["A"_i , " X = ",
  MatrixForm[A_i] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"_i, " = [A,B] is"];
Print["  ", "M"_i, " = ", MatrixForm[M_1]];
Print["The row reduced echelon form for ", "M"_i, " is"];
Print["     ", MatrixForm[ RM_i ]];
```

For the eigenvalue $\lambda_2 = 2$

Solve the equation A_2 X = 0

$$A_2\ X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_2 = [A,B] is

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_2 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_2 = \begin{pmatrix} t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$\lambda_2 = 2$

$$V_2 = \begin{pmatrix} t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix}$$

Does $A\ V_2 = \lambda_2 V_2$?

$$A\ V_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \\ -2t \\ 0 \\ 2t \end{pmatrix}$$

$$\lambda_2 V_2 = 2 \begin{pmatrix} t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \\ -2t \\ 0 \\ 2t \end{pmatrix}$$

True

Investigate the eigen-pair λ_3, V_3

```
i = 3;
Aᵢ = ReplaceAll[M, λ → λᵢ];
vars = {x₁ , x₂ , x₃, x₄ , x₅};
B = {0, 0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mᵢ = AppendRows[Aᵢ, Partition[B, 1]];
RMᵢ = RowReduce[ Mᵢ];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"ᵢ, " = ", λᵢ];
Print["Solve the equation ", "A"ᵢ , " X = 0"];
Print["A"ᵢ , " X = ",
  MatrixForm[Aᵢ] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"ᵢ, " = [A,B] is"];
Print["  ", "M"ᵢ, " = ", MatrixForm[M₁]];
Print["The row reduced echelon form for ", "M"ᵢ, " is"];
Print["     ", MatrixForm[ RMᵢ ]];
```

For the eigenvalue $\lambda_3 = 3$

Solve the equation A_3 X = 0

$$A_3\ X = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_3 = [A,B] is

$$M_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -t \\ 0 & 1 & 0 & 0 & 0 & -t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_3 = \begin{pmatrix} -t \\ -t \\ 0 \\ t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"_i, " = ", λ_i];
Print["V"_i, " = ", MatrixForm[V_i]];
Print["Does  ", "A ", "V"_i, " = ", "λ"_i, "V"_i, " ?"];
Print["A ", "V"_i, " = ", MatrixForm[A],
  MatrixForm[V_i], " = ", MatrixForm[A.V_i]];
Print["λ"_i, "V"_i, " = ", λ_i, MatrixForm[V_i], " = ", MatrixForm[λ_i V_i]];
Print[ExpandAll[A.V_i == λ_i V_i]];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$\lambda_3 = 3$

$$V_3 = \begin{pmatrix} -t \\ -t \\ 0 \\ t \\ t \end{pmatrix}$$

Does $A\, V_3 = \lambda_3 V_3$?

$$A\, V_3 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -t \\ -t \\ 0 \\ t \\ t \end{pmatrix} = \begin{pmatrix} -3t \\ -3t \\ 0 \\ 3t \\ 3t \end{pmatrix}$$

$$\lambda_3 V_3 = 3 \begin{pmatrix} -t \\ -t \\ 0 \\ t \\ t \end{pmatrix} = \begin{pmatrix} -3t \\ -3t \\ 0 \\ 3t \\ 3t \end{pmatrix}$$

True

Investigate the eigen-pair λ_4, V_4

```
i = 4;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4 , x5};
B = {0, 0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue   ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_4 = 2 - \sqrt{3}$

Solve the equation A_4 X = 0

$$A_4\ X = \begin{pmatrix} \sqrt{3} & 1 & 0 & 0 & 0 \\ 1 & \sqrt{3} & 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} & 1 & 0 \\ 0 & 0 & 1 & \sqrt{3} & 1 \\ 0 & 0 & 0 & 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_4 = [A,B] is

$$M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_4 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & 0 & -\sqrt{3}\, t \\ 0 & 0 & 1 & 0 & 0 & 2\, t \\ 0 & 0 & 0 & 1 & 0 & -\sqrt{3}\, t \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_4 = \begin{pmatrix} t \\ -\sqrt{3}\, t \\ 2\, t \\ -\sqrt{3}\, t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_4 = 2 - \sqrt{3}$$

$$V_4 = \begin{pmatrix} t \\ -\sqrt{3}\, t \\ 2t \\ -\sqrt{3}\, t \\ t \end{pmatrix}$$

Does A V_4 = $\lambda_4 V_4$?

$$A\ V_4 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} t \\ -\sqrt{3}\, t \\ 2t \\ -\sqrt{3}\, t \\ t \end{pmatrix} = \begin{pmatrix} 2t - \sqrt{3}\, t \\ 3t - 2\sqrt{3}\, t \\ 4t - 2\sqrt{3}\, t \\ 3t - 2\sqrt{3}\, t \\ 2t - \sqrt{3}\, t \end{pmatrix}$$

$$\lambda_4 V_4 = 2 - \sqrt{3} \begin{pmatrix} t \\ -\sqrt{3}\, t \\ 2t \\ -\sqrt{3}\, t \\ t \end{pmatrix} = \begin{pmatrix} \left(2 - \sqrt{3}\right) t \\ -\sqrt{3} \left(2 - \sqrt{3}\right) t \\ 2\left(2 - \sqrt{3}\right) t \\ -\sqrt{3} \left(2 - \sqrt{3}\right) t \\ \left(2 - \sqrt{3}\right) t \end{pmatrix}$$

True

Investigate the eigen-pair λ_5, V_5

```
i = 5;
Ai = ReplaceAll[M, λ → λi];
vars = {x1 , x2 , x3, x4 , x5};
B = {0, 0, 0, 0, 0};
Needs["LinearAlgebra`MatrixManipulation`"];
Mi = AppendRows[Ai, Partition[B, 1]];
RMi = RowReduce[ Mi];
Needs["LinearAlgebra`MatrixManipulation`"];
Print["For the eigenvalue    ", "λ"i, " = ", λi];
Print["Solve the equation ", "A"i , " X = 0"];
Print["A"i , " X = ",
  MatrixForm[Ai] MatrixForm[vars], " = ", MatrixForm[B]];
Print["The augmented matrix ", "M"i, " = [A,B] is"];
Print["  ", "M"i, " = ", MatrixForm[M1]];
Print["The row reduced echelon form for ", "M"i, " is"];
Print["     ", MatrixForm[ RMi ]];
```

For the eigenvalue $\lambda_5 = 2 + \sqrt{3}$

Solve the equation A_5 X = 0

$$A_5\ X = \begin{pmatrix} -\sqrt{3} & 1 & 0 & 0 & 0 \\ 1 & -\sqrt{3} & 1 & 0 & 0 \\ 0 & 1 & -\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 & -\sqrt{3} & 1 \\ 0 & 0 & 0 & 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented matrix M_5 = [A,B] is

$$M_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The row reduced echelon form for M_5 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Introduce the free variables and find the eigenvector.

```
Print["Introduce the free variables"];
FMi = FreeVariables[RMi];
Print[MatrixForm[FMi]];
Print["Find the reduced row echelon form"];
SMi = RowReduce[ FMi];
Print[MatrixForm[SMi]];
Print["The eigenvector is in the last column"];
Vi = TakeColumns[SMi, -1];
Print["V"i, " = ", MatrixForm[Vi]];
```

Introduce the free variables

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

Find the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & 0 & \sqrt{3}\, t \\ 0 & 0 & 1 & 0 & 0 & 2\, t \\ 0 & 0 & 0 & 1 & 0 & \sqrt{3}\, t \\ 0 & 0 & 0 & 0 & 1 & t \end{pmatrix}$$

The eigenvector is in the last column

$$V_5 = \begin{pmatrix} t \\ \sqrt{3}\, t \\ 2\, t \\ \sqrt{3}\, t \\ t \end{pmatrix}$$

Verify the eigenpair.

```
Print["A = ", MatrixForm[A]];
Print["λ"i, " = ", λi];
Print["V"i, " = ", MatrixForm[Vi]];
Print["Does  ", "A ", "V"i, " = ", "λ"i, "V"i, " ?"];
Print["A ", "V"i, " = ", MatrixForm[A],
  MatrixForm[Vi], " = ", MatrixForm[A.Vi]];
Print["λ"i, "V"i, " = ", λi, MatrixForm[Vi], " = ", MatrixForm[λi Vi]];
Print[ExpandAll[A.Vi == λi Vi]];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_5 = 2 + \sqrt{3}$$

$$V_5 = \begin{pmatrix} t \\ \sqrt{3}\, t \\ 2\,t \\ \sqrt{3}\, t \\ t \end{pmatrix}$$

Does $A\ V_5 = \lambda_5 V_5$?

$$A\ V_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} t \\ \sqrt{3}\, t \\ 2\,t \\ \sqrt{3}\, t \\ t \end{pmatrix} = \begin{pmatrix} 2\,t + \sqrt{3}\, t \\ 3\,t + 2\sqrt{3}\, t \\ 4\,t + 2\sqrt{3}\, t \\ 3\,t + 2\sqrt{3}\, t \\ 2\,t + \sqrt{3}\, t \end{pmatrix}$$

$$\lambda_5 V_5 = 2 + \sqrt{3} \begin{pmatrix} t \\ \sqrt{3}\, t \\ 2\,t \\ \sqrt{3}\, t \\ t \end{pmatrix} = \begin{pmatrix} \left(2 + \sqrt{3}\right) t \\ \sqrt{3}\left(2 + \sqrt{3}\right) t \\ 2\left(2 + \sqrt{3}\right) t \\ \sqrt{3}\left(2 + \sqrt{3}\right) t \\ \left(2 + \sqrt{3}\right) t \end{pmatrix}$$

True

The five eigen-pairs are:

```
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ 5, i++,
  Print["λ"i, " = ", λi, ",   ", "V"i, " = ", MatrixForm[Vi]]; ];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = 1, \quad V_1 = \begin{pmatrix} -t \\ t \\ 0 \\ -t \\ t \end{pmatrix}$$

$$\lambda_2 = 2, \quad V_2 = \begin{pmatrix} t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix}$$

$$\lambda_3 = 3, \quad V_3 = \begin{pmatrix} -t \\ -t \\ 0 \\ t \\ t \end{pmatrix}$$

$$\lambda_4 = 2 - \sqrt{3}, \quad V_4 = \begin{pmatrix} t \\ -\sqrt{3}\, t \\ 2\, t \\ -\sqrt{3}\, t \\ t \end{pmatrix}$$

$$\lambda_5 = 2 + \sqrt{3}, \quad V_5 = \begin{pmatrix} t \\ \sqrt{3}\, t \\ 2\, t \\ \sqrt{3}\, t \\ t \end{pmatrix}$$

We can compare this with the results obtained using Mathematicas **Eigensystem** procedure.

```
sol = Eigensystem[A];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"i, " = ", sol[[1,i]], ",   ", "V"i, " = ", MatrixForm[sol[[2,i]]]]; ];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = 2 + \sqrt{3}, \quad V_1 = \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \quad V_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 2, \quad V_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_4 = 1, \quad V_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_5 = 2 - \sqrt{3}, \quad V_5 = \begin{pmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{pmatrix}$$

Example 8. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Solution 8.

```
A = (2 1 0 0 0 0
     1 2 1 0 0 0
     0 1 2 1 0 0
     0 0 1 2 1 0
     0 0 0 1 2 1
     0 0 0 0 1 2);

n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$A - \lambda I_6 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_6 = \begin{pmatrix} 2-\lambda & 1 & 0 & 0 & 0 & 0 \\ 1 & 2-\lambda & 1 & 0 & 0 & 0 \\ 0 & 1 & 2-\lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 2-\lambda & 1 & 0 \\ 0 & 0 & 0 & 1 & 2-\lambda & 1 \\ 0 & 0 & 0 & 0 & 1 & 2-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$p[\lambda] = 7 - 56\,\lambda + 126\,\lambda^2 - 120\,\lambda^3 + 55\,\lambda^4 - 12\,\lambda^5 + \lambda^6$

$p[\lambda] = \left(-7 + 14\,\lambda - 7\,\lambda^2 + \lambda^3\right)\left(-1 + 6\,\lambda - 5\,\lambda^2 + \lambda^3\right)$

To find the eigenvalues of the matrix A

Solve $7 - 56\,\lambda + 126\,\lambda^2 - 120\,\lambda^3 + 55\,\lambda^4 - 12\,\lambda^5 + \lambda^6 == 0$

Get

$$\lambda_1 = \frac{7}{3} + \frac{7^{2/3}}{3\left(\frac{1}{2}\left(-1 + 3\,i\,\sqrt{3}\right)\right)^{1/3}} + \frac{1}{3}\left(\frac{7}{2}\left(-1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 3.80194$$

$$\lambda_2 = \frac{7}{3} - \frac{\left(\frac{7}{2}\right)^{2/3}\left(1 + i\,\sqrt{3}\right)}{3\left(-1 + 3\,i\,\sqrt{3}\right)^{1/3}} - \frac{1}{6}\left(1 - i\,\sqrt{3}\right)\left(\frac{7}{2}\left(-1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 0.75302$$

$$\lambda_3 = \frac{7}{3} - \frac{\left(\frac{7}{2}\right)^{2/3}\left(1 - i\,\sqrt{3}\right)}{3\left(-1 + 3\,i\,\sqrt{3}\right)^{1/3}} - \frac{1}{6}\left(1 + i\,\sqrt{3}\right)\left(\frac{7}{2}\left(-1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 2.44504$$

$$\lambda_4 = \frac{5}{3} + \frac{7^{2/3}}{3\left(\frac{1}{2}\left(1 + 3\,i\,\sqrt{3}\right)\right)^{1/3}} + \frac{1}{3}\left(\frac{7}{2}\left(1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 3.24698$$

$$\lambda_5 = \frac{5}{3} - \frac{\left(\frac{7}{2}\right)^{2/3}\left(1 + i\,\sqrt{3}\right)}{3\left(1 + 3\,i\,\sqrt{3}\right)^{1/3}} - \frac{1}{6}\left(1 - i\,\sqrt{3}\right)\left(\frac{7}{2}\left(1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 0.198062$$

$$\lambda_6 = \frac{5}{3} - \frac{\left(\frac{7}{2}\right)^{2/3}\left(1 - i\,\sqrt{3}\right)}{3\left(1 + 3\,i\,\sqrt{3}\right)^{1/3}} - \frac{1}{6}\left(1 + i\,\sqrt{3}\right)\left(\frac{7}{2}\left(1 + 3\,i\,\sqrt{3}\right)\right)^{1/3} = 1.55496$$

Symbolic solution of the eigenvectors for this example can be done, however the numerical solution is easier to read.

```
sol = Eigensystem[N[A]];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"i, " = ", sol[[1,i]], ",   ", "V"i, " = ", MatrixForm[sol[[2,i]]]; ];
```

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 = 3.80194, \quad V_1 = \begin{pmatrix} 0.231921 \\ 0.417907 \\ 0.521121 \\ 0.521121 \\ 0.417907 \\ 0.231921 \end{pmatrix}$$

$$\lambda_2 = 3.24698, \quad V_2 = \begin{pmatrix} 0.417907 \\ 0.521121 \\ 0.231921 \\ -0.231921 \\ -0.521121 \\ -0.417907 \end{pmatrix}$$

$$\lambda_3 = 2.44504, \quad V_3 = \begin{pmatrix} 0.521121 \\ 0.231921 \\ -0.417907 \\ -0.417907 \\ 0.231921 \\ 0.521121 \end{pmatrix}$$

$$\lambda_4 = 1.55496, \quad V_4 = \begin{pmatrix} -0.521121 \\ 0.231921 \\ 0.417907 \\ -0.417907 \\ -0.231921 \\ 0.521121 \end{pmatrix}$$

$$\lambda_5 = 0.75302, \quad V_5 = \begin{pmatrix} 0.417907 \\ -0.521121 \\ 0.231921 \\ 0.231921 \\ -0.521121 \\ 0.417907 \end{pmatrix}$$

$$\lambda_6 = 0.198062, \quad V_6 = \begin{pmatrix} -0.231921 \\ 0.417907 \\ -0.521121 \\ 0.521121 \\ -0.417907 \\ 0.231921 \end{pmatrix}$$

Example 9. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 1 & 3 \\ 2 & 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 3 & 4 \end{pmatrix}.$$

Solution 9.

```
A = (5 1 2 0 4
     1 4 2 1 3
     2 2 5 4 0
     0 1 4 1 3
     4 3 0 3 4);

n = Length[A];
In = IdentityMatrix[n];
M = A - λ In;
p[λ_] = Det[A - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"n, " = ", MatrixForm[A], " - λ", MatrixForm[In]];
Print["A - λ", "I"n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ In|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"i, " = ", λi, " = ", Chop[N[λi]]] ];
```

$$A = \begin{pmatrix} 5 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 1 & 3 \\ 2 & 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 3 & 4 \end{pmatrix}$$

$$A - \lambda I_5 = \begin{pmatrix} 5 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 1 & 3 \\ 2 & 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 3 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_5 = \begin{pmatrix} 5-\lambda & 1 & 2 & 0 & 4 \\ 1 & 4-\lambda & 2 & 1 & 3 \\ 2 & 2 & 5-\lambda & 4 & 0 \\ 0 & 1 & 4 & 1-\lambda & 3 \\ 4 & 3 & 0 & 3 & 4-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda I_n|$

$p[\lambda] = -1222 + 1153\lambda - 146\lambda^2 - 79\lambda^3 + 19\lambda^4 - \lambda^5$

To find the eigenvalues of the matrix A

Solve $-1222 + 1153\lambda - 146\lambda^2 - 79\lambda^3 + 19\lambda^4 - \lambda^5 == 0$

Get

$\lambda_1 = \text{Root}[1222 - 1153\,\#1 + 146\,\#1^2 + 79\,\#1^3 - 19\,\#1^4 + \#1^5\,\&, 1] = -3.55784$

$\lambda_2 = \text{Root}[1222 - 1153\,\#1 + 146\,\#1^2 + 79\,\#1^3 - 19\,\#1^4 + \#1^5\,\&, 2] = 1.49766$

$\lambda_3 = \text{Root}[1222 - 1153\,\#1 + 146\,\#1^2 + 79\,\#1^3 - 19\,\#1^4 + \#1^5\,\&, 3] = 3.36188$

$\lambda_4 = \text{Root}[1222 - 1153\,\#1 + 146\,\#1^2 + 79\,\#1^3 - 19\,\#1^4 + \#1^5\,\&, 4] = 5.67255$

$\lambda_5 = \text{Root}[1222 - 1153\,\#1 + 146\,\#1^2 + 79\,\#1^3 - 19\,\#1^4 + \#1^5\,\&, 5] = 12.0258$

Mathematica did not make a mistake ! We just don't know what it meant ! Just ask it for those numbers !

```
N[solset]
```

```
{{λ → -3.55784}, {λ → 1.49766},
 {λ → 3.36188}, {λ → 5.67255}, {λ → 12.0258}}
```

We could have avoided the problem completely by just using the numerical features of *Mathematica*.

```
NSolve[p[λ] == 0]
```

```
{{λ → -3.55784}, {λ → 1.49766},
 {λ → 3.36188}, {λ → 5.67255}, {λ → 12.0258}}
```

```
Eigenvalues[N[A]]
```

```
{12.0258, 5.67255, -3.55784, 3.36188, 1.49766}
```

Symbolic solution of the eigenvectors for this example can be done, however the numerical solution is easier to read.

```
sol = Eigensystem[N[A]];
n = Length[A];
Print["A = ", MatrixForm[A]];
For[i = 1, i ≤ n, i++,
  Print["λ"i, " = ", sol[[1,i]], ",   ", "V"i, " = ", MatrixForm[sol[[2,i]]]]; ];
```

$$A = \begin{pmatrix} 5 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 1 & 3 \\ 2 & 2 & 5 & 4 & 0 \\ 0 & 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 3 & 4 \end{pmatrix}$$

$$\lambda_1 = 12.0258, \quad V_1 = \begin{pmatrix} -0.485108 \\ -0.411359 \\ -0.450676 \\ -0.343352 \\ -0.523884 \end{pmatrix}$$

$$\lambda_2 = 5.67255, \quad V_2 = \begin{pmatrix} -0.440209 \\ 0.0115005 \\ 0.712073 \\ 0.334081 \\ -0.432927 \end{pmatrix}$$

$$\lambda_3 = -3.55784, \quad V_3 = \begin{pmatrix} 0.311869 \\ 0.182146 \\ -0.425024 \\ 0.662313 \\ -0.500256 \end{pmatrix}$$

$$\lambda_4 = 3.36188, \quad V_4 = \begin{pmatrix} 0.616027 \\ -0.690171 \\ 0.2933 \\ -0.0830438 \\ -0.22639 \end{pmatrix}$$

$$\lambda_5 = 1.49766, \quad V_5 = \begin{pmatrix} -0.306813 \\ -0.56669 \\ -0.152226 \\ 0.570039 \\ 0.486427 \end{pmatrix}$$

Exact equation solving

1. For linear equations Gaussian elimination and other methods of linear algebra are used.

2. **Root** objects representing algebraic numbers are usually isolated and manipulated using validated numerical methods.

With **ExactRootIsolation->True**, **Root** uses for real roots a continued fraction version of an algorithm based on Descartes' rule of signs,

and for complex roots the Collins-Krandick algorithm.

3. For single polynomial equations, **Solve** uses explicit formulas up to degree four,attempts to reduce polynomials using **Factor** and **Decompose**,

and recognizes cyclotomic and other special polynomials.

4. For systems of polynomial equations, **Solve** constructs a Gröbner basis.

5. **Solve** and **GrobnerBasis** use an efficient version of the Buchberger algorithm.

6. For non-polynomial equations, **Solve** attempts to change variables and add polynomial side conditions.

7. The code inside *Mathematica* for **Solve** is about **500 pages long**.

Comments. How much do we know or teach about polynomials ? How much should we ? The area of computer science called artificial intelligence treats the topic of "expert system." To construct an "expert system" you tap into the brains of the experts, and program them into your computer. Apparently *Mathematica* has already done this, it took 500 lines of code. They speak of "information overload" in the future. The future was 13 years ago with *Mathematica* ©1988 and 20 years ago with Maple ©1981.

Example 10. Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

Solution 10.

$$A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2}
\end{pmatrix};$$

```
n = Length[A];
I_n = IdentityMatrix[n];
M = A - λ I_n;
p[λ_] = Det[A - λ I_n];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λ_i = solset[[i,1,2]];] ;
Print["      A = ", MatrixForm[A]];
Print["A - λ", "I"_n, " = ", MatrixForm[A], " - λ", MatrixForm[I_n]];
Print["A - λ", "I"_n, " = ", MatrixForm[M]];
Print["The characteristic polynomial is"];
Print["p[λ] = |A-λ I_n|"];
Print["p[λ] = ", p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print["p[λ] = ", q[λ]]];
Print["To find the eigenvalues of the matrix A"];
Print["Solve  ", p[λ] == 0];
Print["Get"];
For[i = 1, i ≤ n, i++,
  Print["  ", "λ"_i, " = ", Chop[N[λ_i]]] ];
```

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

$$A - \lambda I_{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda I_{11} = \begin{pmatrix} \frac{1}{2}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2}-\lambda \end{pmatrix}$$

The characteristic polynomial is

$p[\lambda] = |A-\lambda\ I_n|$

$$p[\lambda] = \frac{11\lambda}{39366} - \frac{13\lambda^2}{78732} - \frac{145\lambda^3}{8748} + \frac{7\lambda^4}{243} + \frac{113\lambda^5}{486} - \frac{679\lambda^6}{972} - \frac{13\lambda^7}{36} + \frac{415\lambda^8}{108} - \frac{217\lambda^9}{36} + 4\lambda^{10} - \lambda^{11}$$

$$p[\lambda] = -\frac{1}{78732}(-1+\lambda)\lambda \left(-11+12\lambda+135\lambda^2-297\lambda^3+162\lambda^4\right)\left(-2-3\lambda+90\lambda^2+27\lambda^3-567\lambda^4+486\lambda^5\right)$$

To find the eigenvalues of the matrix A

Solve $$\frac{11\lambda}{39366} - \frac{13\lambda^2}{78732} - \frac{145\lambda^3}{8748} + \frac{7\lambda^4}{243} + \frac{113\lambda^5}{486} - \frac{679\lambda^6}{972} - \frac{13\lambda^7}{36} + \frac{415\lambda^8}{108} - \frac{217\lambda^9}{36} + 4\lambda^{10} - \lambda^{11} == 0$$

Get

$\lambda_1 = 0$

$\lambda_2 = 1.$

$\lambda_3 = 0.742423$

$\lambda_4 = 0.96944$

$\lambda_5 = -0.252363$

$\lambda_6 = 0.373834$

$\lambda_7 = -0.313629$

$\lambda_8 = -0.146901$

$\lambda_9 = 0.178589$

$\lambda_{10} = 0.567888$

λ_{11} = 0.880719

```
solset2 = Eigenvalues[N[A]];
Sort[Chop[solset2], Less]
```

```
{-0.313629, -0.252363, -0.146901, 0, 0.178589,
 0.373834, 0.567888, 0.742423, 0.880719, 0.96944, 1.}
```

Are the answers the same ? It might be good to sort the answers for comparison.

```
Sort[Chop[N[solset[[All,1,2]]]], Less]
```

```
{-0.313629, -0.252363, -0.146901, 0, 0.178589,
 0.373834, 0.567888, 0.742423, 0.880719, 0.96944, 1.}
```

Of course they are the same (when $n \le 20$).

Mathematica has a very accurate NSolve, Eigenvalues subroutine. We expect them to agree. But root finding algorithms have their limitations.

Example 11. Find the eigenvalues of the following matrix **A**.

Solution 11.

```
n = 25;
A = DiagonalMatrix[Table[1/3, {n}]];
For[i = 1, i ≤ n - 1, i++,
  A[[i,i+1]] = A[[i+1,i]] = 1/3;];
A[[1,1]] = A[[2,1]] = A[[n-1,n]] = A[[n,n]] = 1/2;
Print[MatrixForm[A]];
Id = IdentityMatrix[n];
Clear[p, λ];
p[λ_] = Det[A - Id λ];
Print["p[λ] = ", p[λ] ];
solset1 = NSolve[p[λ] == 0];
solset2 = Eigenvalues[N[A]];
Print["Solution for the eigenvalues"];
Print["By using Root Finding and solving p[λ] = 0"];
Print["By using Mathematica's procedure `Eigenvalues`"];
Print[TableForm[
   Transpose[{Sort[solset1[[All,1,2]], Greater], Sort[solset2, Greater]}],
   TableHeadings → {None, {"Root Finding", "Eigenvalue Procedure"}}]];
```

$$
\begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2}
\end{pmatrix}
$$

$$
\begin{aligned}
p[\lambda] = {} & \frac{1}{376\,572\,715\,308} \Big(1 - 25\,\lambda - 924\,\lambda^2 + 8796\,\lambda^3 + 131\,634\,\lambda^4 - 1\,010\,394\,\lambda^5 - 6\,395\,760\,\lambda^6 + \\
& 54\,033\,480\,\lambda^7 + 108\,315\,549\,\lambda^8 - 1\,488\,881\,169\,\lambda^9 + 659\,577\,330\,\lambda^{10} + \\
& 20\,887\,402\,770\,\lambda^{11} - 46\,521\,636\,552\,\lambda^{12} - 115\,605\,423\,612\,\lambda^{13} + \\
& 586\,573\,752\,222\,\lambda^{14} - 358\,145\,530\,074\,\lambda^{15} - 2\,464\,113\,884\,265\,\lambda^{16} + \\
& 6\,298\,553\,169\,999\,\lambda^{17} - 3\,192\,215\,689\,197\,\lambda^{18} - 12\,411\,790\,206\,093\,\lambda^{19} + \\
& 31\,615\,836\,425\,334\,\lambda^{20} - 37\,959\,459\,512\,220\,\lambda^{21} + 27\,660\,660\,653\,133\,\lambda^{22} - \\
& 12\,521\,042\,783\,991\,\lambda^{23} + 3\,263\,630\,199\,336\,\lambda^{24} - 376\,572\,715\,308\,\lambda^{25}\Big)
\end{aligned}
$$

Solution for the eigenvalues

```
By using Root Finding and solving p[λ] = 0

By using Mathematica's procedure `Eigenvalues`
```

Greater::nord : Invalid comparison with $0.950608 - 1.40205 \times 10^{-7}\, i$ attempted. ≫

Greater::nord : Invalid comparison with $0.977945 + 1.5117 \times 10^{-6}\, i$ attempted. ≫

Greater::nord : Invalid comparison with $0.977945 + 1.5117 \times 10^{-6}\, i$ attempted. ≫

General::stop : Further output of Greater::nord will be suppressed during this calculation. ≫

Root Finding	Eigenvalue Procedure
0.913129	1.
0.599452	0.994452
$1.00003 - 1.27172 \times 10^{-6}\, i$	0.977902
0.865962	0.950632
0.519009	0.913107
$0.977945 + 1.5117 \times 10^{-6}\, i$	0.865966
0.810014	0.810013
0.435647	0.7462
$0.950608 - 1.40205 \times 10^{-7}\, i$	0.675613
0.7462	0.599452
0.350775	0.519009
$0.994735 + 0.0000959837\, i$	0.435647
0.675614	0.350775
0.265821	0.265821
0.182208	0.182208
0.101323	0.101323
0.0244927	0.0244927
−0.0470496	−0.0470496
−0.112192	−0.112192
−0.169975	−0.169975
−0.219624	−0.219624
−0.260575	−0.260575
−0.292487	−0.292487
−0.315227	−0.315227
−0.328817	−0.328817

Which method do we trust ?

Are the answers on the left better or are the answers on right better ?

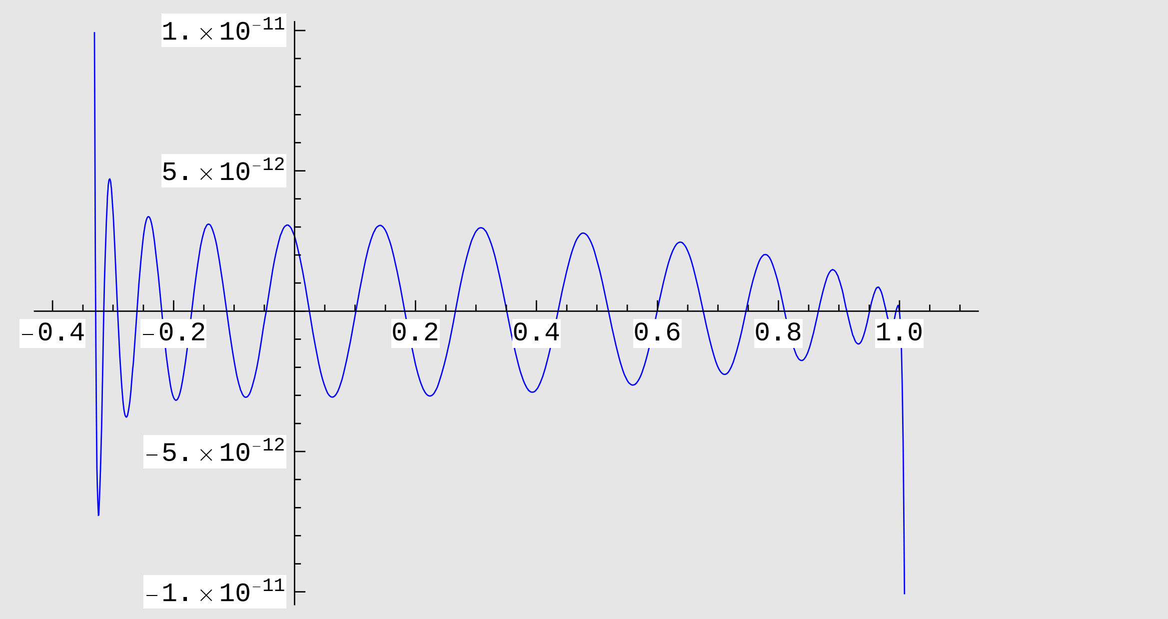
We trust *Mathematica*'s Eigenvalue subroutine !

We should go with the answers on the right. This is a transition matrix and it is known that one of its eigenvalues is $\lambda = 1$.

The eigenvalue-eigenvector methods we study seem overbearing, but they are necessary when n > 20.

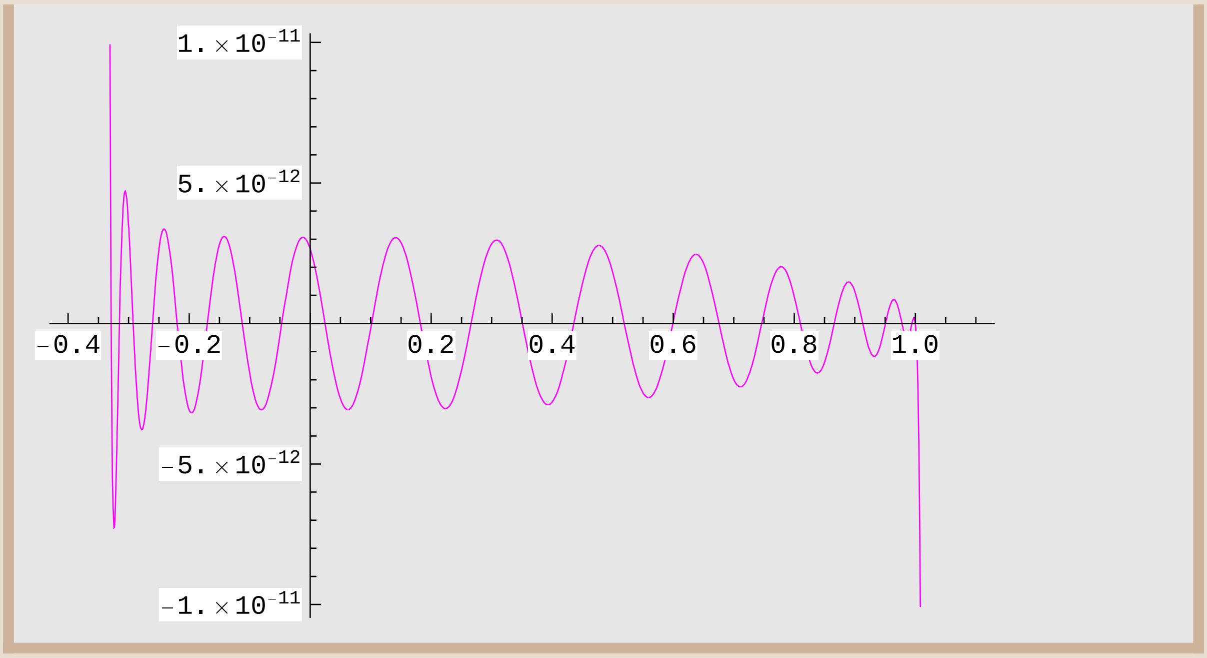
We could try to envision the difficulties of finding the root of

```
Plot[p[λ], {λ, -0.4`, 1.1`}, PlotStyle → Blue]
Print["p[λ] = ", p[λ]];
```



$$p[\lambda] = \frac{1}{376572715308}\left(1 - 25\lambda - 924\lambda^2 + 8796\lambda^3 + 131634\lambda^4 - 1010394\lambda^5 - 6395760\lambda^6 + 54033480\lambda^7 + 108315549\lambda^8 - 1488881169\lambda^9 + 659577330\lambda^{10} + 20887402770\lambda^{11} - 46521636552\lambda^{12} - 115605423612\lambda^{13} + 586573752222\lambda^{14} - 358145530074\lambda^{15} - 2464113884265\lambda^{16} + 6298553169999\lambda^{17} - 3192215689197\lambda^{18} - 12411790206093\lambda^{19} + 31615836425334\lambda^{20} - 37959459512220\lambda^{21} + 27660660653133\lambda^{22} - 12521042783991\lambda^{23} + 3263630199336\lambda^{24} - 376572715308\lambda^{25}\right)$$

```
f[x_] = 376572715308 p[x];
Plot[p[x], {x, -0.4, 1.1}, PlotStyle → Magenta]
Print["f[x] = ", f[x]];
NewtonRaphson[1.01, 50]
```



$f[x] = 1 - 25x - 924x^2 + 8796x^3 + 131634x^4 - 1010394x^5 - 6395760x^6 + 54033480x^7 + 108315549x^8 - 1488881169x^9 + 659577330x^{10} + 20887402770x^{11} - 46521636552x^{12} - 115605423612x^{13} + 586573752222x^{14} - 358145530074x^{15} - 2464113884265x^{16} + 6298553169999x^{17} - 3192215689197x^{18} - 12411790206093x^{19} + 31615836425334x^{20} - 37959459512220x^{21} + 27660660653133x^{22} - 12521042783991x^{23} + 3263630199336x^{24} - 376572715308x^{25}$

$p_0 = 1.0100000000000000$, $f[p_0] = -5.849834442138672$

$p_1 = 1.0061548412994270$, $f[p_1] = -1.847553253173828$

$p_2 = 1.0033038861781030$, $f[p_2] = -0.5530929565429687$

$p_3 = 1.0014081138797720$, $f[p_3] = -0.1484947204589844$

$p_4 = 1.0003931787655360$, $f[p_4] = -0.03795623779296875$

$p_5 = 0.9999754664887750$, $f[p_5] = 0.005970001220703125$

$p_6 = 1.0000584127044340$, $f[p_6] = -0.00452423095703125$

$p_7 = 0.9999987101660400$, $f[p_7] = 0.00206756591796875$

$p_8 = 1.0000270760562970$, $f[p_8] = -0.001781463623046875$

$p_9 = 1.0000030318916460$, $f[p_9] = -0.00293731689453125$

$p_{10} = 0.9999629095361780$, $f[p_{10}] = -0.00152587890625$

$p_{11} = 0.9999415669691970$, $f[p_{11}] = 0.005107879638671875$

$p_{12} = 1.0000139532357220$, $f[p_{12}] = -0.00269317626953125$

$p_{13} = 0.9999773095559570$, $f[p_{13}] = 0.004642486572265625$

$p_{14} = 1.0000417756390640$, $f[p_{14}] = 0.000774383544921875$

$p_{15} = 1.0000521301784940$, $f[p_{15}] = -0.003643035888671875$

p_{16} = 1.0000037898855080, $f[p_{16}]$ = −0.004657745361328125

p_{17} = 0.9999400105376060, $f[p_{17}]$ = 0.004871368408203125

p_{18} = 1.0000091781431180, $f[p_{18}]$ = −0.00067901611328125

p_{19} = 0.9999999210411520, $f[p_{19}]$ = 0.003147125244140625

p_{20} = 1.0000430791112200, $f[p_{20}]$ = −0.002613067626953125

p_{21} = 1.0000081915337640, $f[p_{21}]$ = −0.000759124755859375

p_{22} = 0.9999978481543580, $f[p_{22}]$ = −0.00130462646484375

p_{23} = 0.9999799527389340, $f[p_{23}]$ = 0.007465362548828125

p_{24} = 1.0000835269982370, $f[p_{24}]$ = −0.01013946533203125

p_{25} = 0.9999508019828980, $f[p_{25}]$ = 0.007587432861328125

p_{26} = 1.0000579151133200, $f[p_{26}]$ = −0.00455474853515625

p_{27} = 0.9999976555186190, $f[p_{27}]$ = 0.006511688232421875

p_{28} = 1.0000869313706470, $f[p_{28}]$ = −0.01328659057617187

p_{29} = 0.9999139575923390, $f[p_{29}]$ = 0.0106353759765625

p_{30} = 1.0000672624671260, $f[p_{30}]$ = 0.000301361083984375

p_{31} = 1.0000712262674940, $f[p_{31}]$ = 0.003063201904296875

p_{32} = 1.0001114586634400, $f[p_{32}]$ = −0.002689361572265625

p_{33} = 1.0000769577473310, $f[p_{33}]$ = 0.002223968505859375

p_{34} = 1.0001060054992450, $f[p_{34}]$ = −0.004718780517578125

p_{35} = 1.0000452166297500, $f[p_{35}]$ = 0.004364013671875

p_{36} = 1.0001034002589120, $f[p_{36}]$ = −0.001438140869140625

p_{37} = 1.0000848889710520, $f[p_{37}]$ = −0.005596160888671875

p_{38} = 1.0000119852584200, $f[p_{38}]$ = −0.004322052001953125

p_{39} = 0.9999531403919770, $f[p_{39}]$ = 0.004207611083984375

p_{40} = 1.0000124686178840, $f[p_{40}]$ = 0.00580596923828125

p_{41} = 1.0000913673123750, $f[p_{41}]$ = −0.007091522216796875

p_{42} = 0.9999989316980810, $f[p_{42}]$ = −0.00222015380859375

p_{43} = 0.9999684975629880, $f[p_{43}]$ = 0.002674102783203125

p_{44} = 1.0000057650056390, $f[p_{44}]$ = −0.000827789306640625

p_{45} = 0.9999944488525560, $f[p_{45}]$ = 0.004444122314453125

p_{46} = 1.0000555282454900, $f[p_{46}]$ = −0.007320404052734375

p_{47} = 0.9999582382903970, $f[p_{47}]$ = 0.001178741455078125

p_{48} = 0.9999748183899850, $f[p_{48}]$ = 0.001140594482421875

p_{49} = 0.9999907124730910, $f[p_{49}]$ = 0.001209259033203125

p_{50} = 1.0000073534101740, $f[p_{50}]$ = 0.00337982177734375

p = 1.000007353410174

$\triangle p$ = ±0.0000166409

$f[p]$ = 0.00337982177734375

Clearly, there are computational problems in evaluating the Newton-Raphson iteration function. This is why there are methods for finding eigenvalues that do not rely on finding roots of polynomials.